Problem Sheet 2

Notations & Conventions

Greek indices: $\{\mu, \nu, \rho\sigma, \ldots\}$ will denote spacetime indices and Latin indices: $\{i, j, k, l, \ldots\}$ will denote spatial indices. We will work with mostly minus metric convention as in the lectures: $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. We will set $\hbar = c = 1$, mostly.

Questions

1. Consider the real scalar field and its equal time commutation relations:

$$[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = 0, \tag{1}$$

$$[\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0, \tag{2}$$

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}). \tag{3}$$

   a) Show that it is enough to impose the canonical commutation relations at $t = 0$. That is, once we impose them at $t = 0$, then the relations at general t are automatically satisfied.

   Hint: Think of the commutator as an operator itself: $\hat{C}(t, \vec{x}) \equiv [\hat{A}(t, \vec{x}), \hat{B}(t, \vec{x})]$. Let $\hat{A}$ and $\hat{B}$ be scalar operators. Consider the above commutator relation at $t = 0$ and relate it to generic time t relation using $\hat{U}^\dagger(\Lambda)\, \hat{\phi}(x)\, \hat{U}(\Lambda) = \hat{\phi}(\Lambda^{-1}x)$.

   Note: *This is exercise 26 in the lecture notes.*

2. Consider translation generators P_μ and a field operator $\hat{\phi}(x)$. Let the unitary operator implementing a spacetime translation by a small four–vector a^μ be

$$U(a) \equiv e^{\,ia^\mu P_\mu}, \qquad U(a)^\dagger = e^{-\,ia^\mu P_\mu}.$$

   The translated field satisfies

$$U(a)\, \phi(x)\, U(a)^\dagger \;=\; \phi(x + a). \tag{4}$$

   a) Expand the right–hand side to first order in a^μ and write it in index–free form.

Then use the following identity:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots .$$

b) Expanding the exponential for small A and keeping the order of operations right you should be able to see the above identity. Show it. This is called the adjoint form of the Baker–Campbell–Hausdorff expansion.

c) Take $A = ia^\mu P_\mu$ and $B = \phi(x)$ in the formula above and expand $U(a)\phi(x)U(a)^\dagger$ to first order in a^μ.

d) Read off the commutator $[P_\mu, \phi(x)]$ and show that you get:

$$[P_\mu,\, \phi(x)] = -\, i\, \partial_\mu \phi(x) \tag{5}$$

which implies that at the quantum level the momentum operator acts on fields via commutators to generate translations.

Note: *This is exercise 27 in the lecture notes.*

3. Consider the following:

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] \tag{6}$$

where γ^μ for $\mu = 0, 1, 2, 3$ are the generators of the Clifford algebra satisfying:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\, \mathbb{1}, \tag{7}$$

a) Show that $S^{\rho\sigma}$ satisfy the Lorentz algebra.

Now consider a spinor field which under a proper orthochronous Lorentz trnasformations behaves as:

$$\psi'(x) = \mathcal{S}[\Lambda]\, \psi(\Lambda^{-1}x) \tag{8}$$

where $\mathcal{S}[\Lambda]$ is a spinor matrix representation satisfying:

$$\mathcal{S}[\Lambda]^{-1}\gamma^\mu \mathcal{S}[\Lambda] = \Lambda^\mu{}_\nu \gamma^\nu. \tag{9}$$

For an infinitesimal Lorentz transformation we have:

$$\mathcal{S}[\Lambda] = \exp\left(\frac{1}{2}\omega_{\rho\sigma} S^{\rho\sigma}\right) \tag{10}$$

with $S^{\rho\sigma}$ defined in (6).

b) Using $\delta\psi(x) = \frac{1}{2}\omega_{\rho\sigma}(L^{\rho\sigma}\psi)(x)$, show that,

$$L^{\rho\sigma} = S^{\rho\sigma} + x^\sigma \partial^\rho - x^\rho \partial^\sigma. \tag{11}$$

Note: *Part a) above is exercise 31 in the lecture notes.*

4. Consider the massless Dirac action:

$$\mathcal{S}_D = \int d^4x \; \bar{\psi}(x)(i\gamma^\mu \partial_\mu)\psi(x) \tag{12}$$

a) Using the Euler-Lagrange's equations derive its equations of motion.

b) Show that the the above action is invariant under:

$$\psi \;\to\; e^{i\alpha\gamma^5}\psi, \qquad \bar{\psi} \;\to\; \bar{\psi}e^{i\alpha\gamma^5}.$$

where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\alpha \in \mathbb{R}$ is a constant.

c) Find the corresponding conserved Noether current. Denote it by j^μ_A – called the *axial current*.

d) Above you would have gotten: $\partial_\mu j^\mu_A = 0$. What happens to $\partial_\mu j^\mu_A$ if you would have considered the massive Dirac action instead?

Hint: You will find in the massive case: $\partial_\mu j^\mu_a \propto m$. Find the exact expression for the divergence of the axial current.

5. Consider plane-wave solutions to the Dirac equation

$$\psi(x) = u(p)\, e^{-ip\cdot x} = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} e^{-ip\cdot x}$$

Put them into the massless Dirac equation (note that, $p^0 = |\vec{p}|$ when $m = 0$).

a) Show that u_- and u_+ are eigenvectors of the helicity operator

$$\hat{h} := \tfrac{1}{2}\begin{pmatrix} \hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \hat{p}\cdot\vec{\sigma} \end{pmatrix}, \tag{13}$$

where $\hat{p}$ is the unit vector in the direction of $\vec{p}$.

Hint: Work in the Weyl/chiral basis and consider the Weyl equations for u_{pm} (Eq.(7.49) of the typed notes). Without loss of generality, you can also take $\hat{p}$ to lie in the z-direction.

b) List the eigenvalues of the left- (u_-) and right- (u_+)handed spinors.

Note: *This is exercise 34 in the lecture notes.*

6. In this exercise we shall understand a very nice theorem called *Haag's theorem.*

Consider two of real scalar field operators $\hat{\Phi}_1(x)$ and $\hat{\Phi}_2(x)$, with masses m_1 and m_2, respectively, and with corresponding creation and annihilation operators $\hat{a}^\dagger_{1,2}(0,\vec{k})$ and $\hat{a}_{1,2}(0,\vec{k})$. Choose the fields and their conjugate momenta $\hat{\Pi}_{1,2}(t,\vec{x}) = \partial_t\hat{\Phi}_{1,2}(t,\vec{x})$ to have the same initial conditions at $t = 0$, i.e.,

$$\hat{\Phi}_1(0,\vec{x}) = \hat{\Phi}_2(0,\vec{x}) \equiv \hat{\Phi}(0,\vec{x}), \quad \hat{\Pi}_1(0,\vec{x}) = \hat{\Pi}_2(0,\vec{x}) \equiv \hat{\Pi}(0,\vec{x}). \quad (14)$$

a) Show that the creation and annihilation operators of the two fields are related via

$$\hat{a}_2(0,\vec{k}) = \cosh\theta\ \hat{a}_1(0,\vec{k}) \ + \ \sinh\theta\ \hat{a}^\dagger_1(0,\vec{k}), \quad (15)$$
$$\hat{a}_1(0,\vec{k}) = \cosh\theta\ \hat{a}_2(0,\vec{k}) \ - \ \sinh\theta\ \hat{a}^\dagger_2(0,\vec{k}), \quad (16)$$

and determine θ.

Hints:

i) Write down general mode expansions of $\hat{\Phi}_i(t,\vec{x})$ and $\hat{\Pi}_i(t,\vec{x})$ in terms of ladder operators and put $t = 0$ in them.

ii) Invert the above expressions to write $\hat{a}_i(0,\vec{k})$ and $\hat{a}^\dagger_i(0,\vec{k})$ in terms of $\hat{\Phi}_i(0,\vec{x})$ and $\hat{\Pi}_i(0,\vec{x})$.

iii) Put $i = 2$ in above expression and use i) above to express $\hat{a}_2(0,\vec{k})$ in terms of $\hat{\Phi}_1(0,\vec{x})$ and $\hat{\Pi}_1(0,\vec{x})$. Now use (14) to set $\hat{\Phi}_1(0,\vec{x}) = \hat{\Phi}(0,\vec{x})$ and $\hat{\Pi}_1(0,\vec{x}) = \hat{\Pi}(0,\vec{x})$.

iv) Next use expressions for $\hat{\Phi}_1(0,\vec{x})$ and $\hat{\Pi}_1(0,\vec{x})$ in step iii) so as to get an expression for $\hat{a}_2(0,\vec{k})$ in terms of $\hat{a}_1(0,\vec{k})$ and $\hat{a}^\dagger_1(0,\vec{k})$. This will give you (15) with the creation part being: $\hat{a}^\dagger_1(0,-\vec{k})$ which you can relabel – via a change of integration variable – to $\hat{a}^\dagger_1(0,\vec{k})$.

b) Consider what this implies for the action of the operator $\hat{a}_2(0,\vec{k})$ on the vacuum state for the operator $\hat{a}_1(0,\vec{k})$, i.e., that state for which $\hat{a}_1(0,\vec{k})\,|0\rangle = 0$.

c) Compute the number operator, evaluated on the vacuum: $\hat{n}_i(0,\vec{k}) = \langle 0|\hat{a}_i^\dagger(0,\vec{k})\hat{a}_i(0,\vec{k})|0\rangle$. for $i = 1, 2$. You will find it to be zero for $i = 1$ but non-zero for $i = 2$!

Hint: You will fine that the 1-vacuum which is annihilated by $\hat{a}_1(0,\vec{k})$ is not a vacuum for the 2-system!

Now let $\hat{\Phi}_2$ be the full, interacting (Heisenberg) field and $\hat{\Phi}_1$ be the free, interaction-picture field, such that the masses m_1 and m_2 also coincide at time $t = 0$.

c) Show that at $t = 0$ then, $\sinh\theta = 0$ and hence the 1-vacuum is also the 2-vacuum.

The mass m_2 will differ from m_1 for times $t > 0$ due to the interactions since the system 2 is interacting but system 1 is free. This is the content behind *Haag's theorem*: interacting and free fields generally live in *unitarily inequivalent* Fock spaces. Since they are not related by unitary transformations, the norm/number of particles in the vacuum changes that is for $t > 0$: $\hat{n}_1(t,\vec{k}) \neq \hat{n}_2(t,\vec{k})$. This also means whenever you have two observers not related by an unitary transformations but observing the same system then the notion of its *vacuum* will not be the same foe both of them.

d) Give an example where the above situation might arise physically.

Hint: Recall, Lorentz transformations were implemented by unitary operators acting on states of Hilbert spaces. What about generic coordinate transformations? [1] What about generic time-dependent settings?

Submit via Google Form!

Deadline: 25/11/2025

[1] Now we are entering the realm of General Relativity (Hawking/Unruh radiation anybody?)