Q1(a): Infinitesimal Lorentz action on a vector field

Note: Below equation numbers will be from v9 (latest version on slack) of the typed lecture notes.

Setup: A spacetime vector field transforms as

$$A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(\Lambda^{-1}x), \qquad \Lambda(\omega) = \exp\big(\omega_{\rho\sigma} L^{\rho\sigma}\big),$$

with antisymmetric parameters $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$. The generators are

$$\big(M^{\rho\sigma}\big)^{\mu}{}_{\nu} = \eta^{\sigma\mu}\delta^{\rho}{}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}{}_{\nu}. \tag{1}$$

To see the above, start from Eq.(2.14) of the typed notes:

$$(M_{\rho\sigma})^{\mu}{}_{\nu} = \eta_{\rho\nu}\eta_{\sigma\mu} - \eta_{\rho\mu}\eta_{\sigma\nu} \tag{2}$$

Now raise $\rho\sigma$ of M by multiplying above with $\eta^{\rho\alpha}\eta^{\sigma\beta}$ to get:

$$(M^{\alpha\beta})_{\mu\nu} = (M_{\rho\sigma})^{\mu}{}_{\nu}\eta^{\rho\alpha}\eta^{\sigma\beta} = \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} \tag{3}$$

Finally raise the μ on M by multiplying above with $\eta^{\mu\lambda}$ to get,

$$(M^{\alpha\beta})^{\lambda}{}_{\nu} = (M^{\alpha\beta})_{\mu\nu}\eta^{\mu\lambda} = \eta^{\beta\lambda}\delta^{\alpha}_{\nu} - \eta^{\alpha\lambda}\delta^{\beta}_{\nu} \tag{4}$$

which is (1).
For infinitesimal transformations,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega_{\rho\sigma}\big(M^{\rho\sigma}\big)^{\mu}{}_{\nu} + \mathcal{O}(\omega^2), \qquad (\Lambda^{-1})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \omega_{\rho\sigma}\big(M^{\rho\sigma}\big)^{\mu}{}_{\nu} + \mathcal{O}(\omega^2).$$

Variation of the field: Let $\delta x^{\alpha} = \omega_{\rho\sigma}\big(M^{\rho\sigma}\big)^{\alpha}{}_{\beta}x^{\beta}$ so that $\Lambda^{-1}x = x - \delta x + \mathcal{O}(\omega^2)$. Expanding to first order,

$$\begin{aligned} A^{\nu}(\Lambda^{-1}x) &= A^{\nu}(x) - \delta x^{\alpha}\partial_{\alpha}A^{\nu}(x) + \mathcal{O}(\omega^2), \\ A'^{\mu}(x) &= \Big(\delta^{\mu}{}_{\nu} + \omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}\Big)\Big(A^{\nu}(x) - \delta x^{\alpha}\partial_{\alpha}A^{\nu}(x)\Big) + \mathcal{O}(\omega^2) \\ &= A^{\mu}(x) + \omega_{\rho\sigma}(M^{\rho\sigma})^{\mu}{}_{\nu}A^{\nu}(x) - \delta x^{\alpha}\partial_{\alpha}A^{\mu}(x) + \mathcal{O}(\omega^2). \end{aligned}$$

Therefore,

$$\delta A^{\mu}(x) = \omega_{\rho\sigma}\Big[(M^{\rho\sigma})^{\mu}{}_{\nu}A^{\nu}(x) - \big(M^{\rho\sigma}\big)^{\alpha}{}_{\beta}x^{\beta}\,\partial_{\alpha}A^{\mu}(x)\Big].$$

Identifying $L^{\rho\sigma}$: By definition, $A'^{\mu}(x) = \exp(\omega_{\rho\sigma}L^{\rho\sigma})A^{\mu}(x) = A^{\mu}(x) + \omega_{\rho\sigma}(L^{\rho\sigma}A)^{\mu} + \mathcal{O}(\omega^2)$. Hence, $\delta A^{\mu} = \omega_{\rho\sigma}\,(L^{\rho\sigma}A)^{\mu}$. Hence

$$(L^{\rho\sigma}A)^{\mu} = (M^{\rho\sigma})^{\mu}{}_{\nu}A^{\nu} - \left(M^{\rho\sigma}\right)^{\alpha}{}_{\beta}x^{\beta}\,\partial_{\alpha}A^{\mu}.$$

Using (1), the *orbital* piece simplifies to

$$-\left(M^{\rho\sigma}\right)^{\alpha}{}_{\beta}x^{\beta}\,\partial_{\alpha} = -(\eta^{\sigma\alpha}x^{\rho}\partial_{\alpha}A^{\mu} - \eta^{\rho\alpha}x^{\sigma}\partial_{\alpha}A^{\mu}) = (x^{\sigma}\partial^{\rho} - x^{\rho}\partial^{\sigma})A^{\mu},$$

so that

$$(L^{\rho\sigma}A)^{\mu} = (M^{\rho\sigma})^{\mu}{}_{\nu}A^{\nu} + \left(x^{\sigma}\partial^{\rho} - x^{\rho}\partial^{\sigma}\right)A^{\mu}.$$

Note: Anyone who derives till above gets full marks!

Operator form and final result: As differential operators, products like $\partial^{\rho}x^{\sigma}$ mean "multiply by x^{σ} then differentiate with respect to x_{ρ}". Acting on a test field f,

$$\partial^{\rho}(x^{\sigma}f) - \partial^{\sigma}(x^{\rho}f) = x^{\sigma}\partial^{\rho}f - x^{\rho}\partial^{\sigma}f,$$

so $x^{\sigma}\partial^{\rho} - x^{\rho}\partial^{\sigma} = \partial^{\rho}x^{\sigma} - \partial^{\sigma}x^{\rho}$ as operators. Therefore,

$$\boxed{L^{\rho\sigma} = M^{\rho\sigma} + \partial^{\rho}x^{\sigma} - \partial^{\sigma}x^{\rho}}$$

with $L^{\rho\sigma} = -L^{\sigma\rho}$, as required. The first term above is the intrinsic spacetime independent piece which later becomes *spin* upon quantisation and the second term above is the spacetime dependent *orbital* piece like: $\vec{r}\times\vec{p}$.

Q2. EoM of free Maxwell field

$$\mathcal{L}_1 = -\frac{1}{2}(\partial_{\mu}A_{\nu})(\partial^{\mu}A^{\nu}) + \frac{1}{2}(\partial_{\mu}A^{\mu})^2, \qquad \mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},$$

with $\epsilon^{0123} = +1$.

(a) EoMs from $\mathcal{L}_1$

Since $\mathcal{L}_1$ depends only on $\partial_{\mu}A_{\nu}$, the Euler-Lagrange equations become:

$$\frac{\partial\mathcal{L}_1}{\partial A_{\nu}} - \partial_{\mu}\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_{\mu}A_{\nu})}\right) = 0, \quad \Rightarrow \quad \partial_{\mu}\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_{\mu}A_{\nu})}\right) = 0.$$

Using $\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu$,, which implies: $\frac{\partial(\partial_\alpha A^\alpha)}{\partial(\partial_\mu A_\nu)} = \eta^{\mu\nu}$, one finds

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_\mu A_\nu)} = -\,\partial^\mu A^\nu + \eta^{\mu\nu}(\partial_\lambda A^\lambda).$$

Hence

$$-\,\partial_\mu \partial^\mu A^\nu + \partial^\nu(\partial_\lambda A^\lambda) = 0 \quad \Longleftrightarrow \quad \boxed{\Box A^\nu - \partial^\nu(\partial_\lambda A^\lambda) = 0,} \qquad \Box \equiv \partial_\mu \partial^\mu.$$

(b) EoMs from $\mathcal{L}_2$

Because $\mathcal{L}_2$ depends on A_ν only through $F_{\alpha\beta}$, Euler-Lagrange equations in this case also become,

$$\partial_\mu \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu A_\nu)} \right) = 0.$$

Chain rule:

$$\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}_2}{\partial F_{\alpha\beta}} \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}.$$

Compute

$$\frac{\partial \mathcal{L}_2}{\partial F_{\alpha\beta}} = -\frac{1}{4}\frac{\partial}{\partial F_{\alpha\beta}}(F_{\rho\sigma}F^{\rho\sigma}) = -\frac{1}{4}\frac{\partial}{\partial F_{\alpha\beta}}(\eta^{\lambda\rho}\eta^{\theta\sigma}\, F_{\rho\sigma}F_{\lambda\theta}) = -\frac{1}{2}\, F^{\alpha\beta}, \quad \frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha.$$

Thus

$$\frac{\partial \mathcal{L}_2}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2}(F^{\mu\nu} - F^{\nu\mu}) = -\,F^{\mu\nu} \quad \Rightarrow \quad \boxed{\partial_\mu F^{\mu\nu} = 0.}$$

Expanding $\partial_\mu F^{\mu\nu} = 0$ gives

$$\Box A^\nu - \partial^\nu(\partial_\lambda A^\lambda) = 0,$$

so the EoMs from $\mathcal{L}_2$ coincide with those from $\mathcal{L}_1$.

Boundary-term relation: Using $F_{\mu\nu}F^{\mu\nu} = 2\big[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\big]$, one finds:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}(\partial_\mu A_\nu)^2 + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu).$$

Now let us focus on the second term above and integrate it by parts to get:

$$\frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) = \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu) - \frac{1}{2}A_\nu \partial_\mu \partial^\nu A^\mu,$$

where the first term since its a total derivative and we get from the second term,

$$-\frac{1}{2}A_\nu\partial_\mu\partial^\nu A^\mu = -\frac{1}{2}A_\nu\partial^\nu\partial_\mu A^\mu = -\frac{1}{2}\partial^\nu(A_\nu\partial_\mu A^\mu) + \frac{1}{2}(\partial_\mu A^\mu)(\partial^\nu A_\nu).$$

Hence

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{1}{2}\partial_\mu(A_\nu\partial^\nu A^\mu) - \frac{1}{2}\partial^\nu(A_\nu\partial_\mu A^\mu)$$

and the actions differ only by a total derivative term.

(c) Canonical momenta and the role of A_0

Canonical momentum: $\pi^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\nu)}$.

From $\mathcal{L}_2$*:* $\pi^i = F^{i0} \equiv E^i$, $\boxed{\pi^0 = 0}$. Thus A_0 is nondynamical (a Lagrange multiplier imposing Gauss's law).

From $\mathcal{L}_1$*:* $\pi^\nu = -\partial^0 A^\nu + \eta^{0\nu}(\partial\cdot A) \Rightarrow \pi^0 = -\partial_i A_i$, $\pi^i = \partial_0 A_i$. Again, no kinetic term for A_0.

(d) Source-free Gauss and Ampère–Maxwell

Identify $F^i{}_0 = E^i$, $F^i{}_j = \epsilon^i{}_{jk}B^k$ (with $\epsilon^{123} = +1$). From $\partial_\mu F^{\mu\nu} = 0$:

$\nu = 0:\quad \partial_i F^{i0} = 0 \Rightarrow \boxed{\nabla\cdot\mathbf{E} = 0}$, $\qquad \nu = i:\quad -\partial_t E^i + (\nabla\times\mathbf{B})^i = 0 \Rightarrow \boxed{\nabla\times\mathbf{B} - \partial_t\mathbf{E} = 0}$.

(e) The remaining pair and covariant form

The other two Maxwell equations are the Bianchi identities, which follow as an (off-shell) identity from the definition: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0}$$

The above equation is called the *Bianchi identity*.

Deriving Faraday's law and Gauss's law for magnetostatics:

Start from the Bianchi identity

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{5}$$

and the identifications

$$F^i{}_0 = E^i, \qquad F^i{}_j = \epsilon^i{}_{jk}B^k, \qquad \epsilon^{123} = +1, \tag{6}$$

so that (lowered indices) $F_{i0} = -E_i$, $F_{0i} = +E_i$, $F_{ij} = \epsilon_{ij}{}^k B_k$ with spatial metric being Euclidean: $\delta_{ij} = \mathrm{diag}(1,1,1)$.

Faraday's law

Choose $(\lambda, \mu, \nu) = (0, i, j)$ in (5):

$$\begin{aligned} \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} &= 0 \\ \partial_0(\epsilon_{ij}{}^k B_k) - \partial_i E_j + \partial_j E_i &= 0 \\ \epsilon_{ij}{}^k\, \partial_0 B_k - \epsilon_{kij}\, (\nabla \times \mathbf{E})^k &= 0 \end{aligned}$$

(using definition of curl: $\partial_i E_j - \partial_j E_i = \epsilon_{kij}(\nabla \times \mathbf{E})^k$ and $\epsilon_{ij}{}^k = \epsilon_{kij}$ (since bringing down a spatial index of a purely spatial object, like ϵ_{ijk}, is done using the purely Euclidean metric: $\delta_{ij} = \mathrm{diag}(1, 1, 1)$)). Thus,

$$(\nabla \times \mathbf{E})^k = -\partial_0 B^k \qquad \Longrightarrow \qquad \boxed{\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0}. \tag{7}$$

Gauss's law for magnetism

Choose all-spatial $(\lambda, \mu, \nu) = (i, j, k)$ in (5):

$$\begin{aligned} \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} &= 0 \\ \epsilon_{jk}{}^\ell\, \partial_i B_\ell + \epsilon_{ki}{}^\ell\, \partial_j B_\ell + \epsilon_{ij}{}^\ell\, \partial_k B_\ell &= 0. \end{aligned} \tag{8}$$

Bonus: Contract with $\frac{1}{2}\,\epsilon^{ijk}$ and use the identity: $\epsilon^{ijk}\epsilon_{jk}{}^\ell = 2\delta^{i\ell}$:

$$\partial_i B^i = 0 \qquad \Longrightarrow \qquad \boxed{\nabla \cdot \mathbf{B} = 0}. \tag{9}$$

Note: If you reach till (8), you will get full grades for this part.

Q3(a): EoM of complex scalar field

Consider

$$\mathcal{L}(\Phi, \Phi^*, \partial\Phi, \partial\Phi^*) \;=\; \partial_\mu \Phi^*\, \partial^\mu \Phi \;-\; m^2 \Phi^* \Phi.$$

Treat Φ and Φ^* as independent fields.

Euler–Lagrange for Φ^*:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} = \partial^\mu \Phi, \qquad \frac{\partial \mathcal{L}}{\partial \Phi^*} = -m^2 \Phi.$$

Hence

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0 \;\Longrightarrow\; \boxed{\partial_\mu \partial^\mu \Phi + m^2 \Phi = 0}.$$

Euler–Lagrange for Φ:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial^\mu \Phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi^*.$$

Thus

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0 \implies \boxed{\partial_\mu \partial^\mu \Phi^* + m^2 \Phi^* = 0}.$$

Writing $\Box \equiv \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$, both equations read

$$\boxed{(\Box + m^2)\Phi = 0, \qquad (\Box + m^2)\Phi^* = 0}.$$

Q4. Poincaré charges and algebra

The Poincaré algebra is

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho}, \tag{10}$$

$$[M^{\mu\nu}, P^\rho] = \eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu, \qquad [P^\mu, P^\nu] = 0. \tag{11}$$

Define spatial rotations, boosts, energy and momentum by

$$J^i = \frac{1}{2} \epsilon^{ijk} M^{jk}, \qquad K^i = M^{i0}, \qquad H = P^0.$$

Note: Here we changed the definition of boost as $K^i = M^{i0}$ which is opposite to what we had in the typed notes: $K^i = M^{0i}$. This is a typo in the notes and you can still use that old definition which will give you $[K^i, H] = -P^i$ and $[K^i, P^j] = -\delta^{ij} H$ instead and you will get full grades for getting the above with this minus sign.

Additionally, since this was a typo in the notes, as per the rule, everyone who has solved this part of the question with the minus sign as above get additional +1 point.

Furthermore, I have also raised jk-index in M while defining J^i. This was another typo in the notes. Here, repeated indices are summed. Additionally, any one who has attempted this question will get +1 due to this typo.

(a) Derivation of the listed commutators

Using (10)–(11):

1) $\underline{[J^i, J^j] = \epsilon^{ijk} J^k}$**:** Start:

$$[J^i, J^j] = \frac{1}{4}\,\epsilon^{imn}\epsilon^{jpq}[M^{mn}, M^{pq}].$$

Insert (10) and note that m, n, p, q are spatial so $\eta^{ab} = -\delta^{ab}$:

$$[M^{mn}, M^{pq}] = -\delta^{np}M^{mq} + \delta^{mp}M^{nq} + \delta^{nq}M^{mp} - \delta^{mq}M^{np}.$$

Hence

$$\begin{aligned}[J^i, J^j] &= \frac{1}{4}\,\epsilon^{imn}\epsilon^{jpq}\Big(- \delta^{np}M^{mq} + \delta^{mp}M^{nq} + \delta^{nq}M^{mp} - \delta^{mq}M^{np}\Big) \\ &= \frac{1}{4}\Big(- \epsilon^{imn}\epsilon^{jnq}M^{mq} + \epsilon^{imn}\epsilon^{jmq}M^{nq} + \epsilon^{imn}\epsilon^{jpn}M^{mp} - \epsilon^{imn}\epsilon^{jpm}M^{np}\Big).\end{aligned}$$

Now, in each term relabel dummies so every piece looks like a single ϵ times M^{rs}, and use antisymmetry $M^{rs} = -M^{sr}$ to align indices:

$$\begin{aligned}&\text{A: } - \epsilon^{imn}\epsilon^{jnq}M^{mq} \xrightarrow{m\to r,\, q\to s} -\epsilon^{irn}\epsilon^{jns}M^{rs}, \\ &\text{B: } \epsilon^{imn}\epsilon^{jmq}M^{nq} \xrightarrow{n\to r,\, q\to s} \epsilon^{irn}\epsilon^{jrs}M^{rs}, \\ &\text{C: } \epsilon^{imn}\epsilon^{jpn}M^{mp} \xrightarrow{m\to r,\, p\to s} \epsilon^{irn}\epsilon^{jsn}M^{rs}, \\ &\text{D: } - \epsilon^{imn}\epsilon^{jpm}M^{np} \xrightarrow{n\to r,\, p\to s} - \epsilon^{irn}\epsilon^{jsm}M^{rs} \quad (\text{rename } m \to n).\end{aligned}$$

Group (A + C) and (B + D); each group is proportional to a single ϵ^{ijk} multiplying M^{rs}. Collecting,

$$[J^i, J^j] = \epsilon^{ijk}\left(\frac{1}{2}\,\epsilon^{krs}M^{rs}\right) = \epsilon^{ijk}\,J^k.$$

2) $\underline{[J^i, K^j] = \epsilon^{ijk} K^k}$**:**

$$\begin{aligned}[J^i, K^j] &= \frac{1}{2}\epsilon^{imn}[M^{mn}, M^{j0}] \\ &= \frac{1}{2}\epsilon^{imn}\Big(\eta^{nj}M^{m0} - \eta^{mj}M^{n0} - \eta^{n0}M^{mj} + \eta^{m0}M^{nj}\Big).\end{aligned}$$

Since $\eta^{n0} = \eta^{m0} = 0$ and $\eta^{nj} = -\delta^{nj}$, $\eta^{mj} = -\delta^{mj}$,

$$[J^i, K^j] = \frac{1}{2}\epsilon^{imn}\big(- \delta^{nj}M^{m0} + \delta^{mj}M^{n0}\big) = \frac{1}{2}\big(- \epsilon^{imj}M^{m0} + \epsilon^{ijn}M^{n0}\big).$$

Rename $m \to k$ and $n \to k$ in the two terms, and note that these two contributions *add*:

$$[J^i, K^j] = \epsilon^{ijk}\,M^{k0} = \epsilon^{ijk}\,K^k.$$

3) $\underline{[K^i, K^j] = -\epsilon^{ijk} J^k}$**:**

$$\begin{aligned}[K^i, K^j] &= [M^{i0}, M^{j0}] \\ &= \eta^{0j} M^{i0} - \eta^{ij} M^{00} - \eta^{00} M^{ij} + \eta^{i0} M^{0j}.\end{aligned}$$

Here $\eta^{0j} = \eta^{i0} = 0,\ M^{00} = 0,\ \eta^{00} = +1,\ \eta^{ij} = -\delta^{ij}$, so

$$[K^i, K^j] = -M^{ij} = -\epsilon^{ijk} J^k.$$

4) $\underline{[J^i, P^j] = \epsilon^{ijk} P^k}$ **and** $\underline{[J^i, H] = 0}$**:**

$$\begin{aligned}[J^i, P^j] &= \frac{1}{2}\epsilon^{imn}[M^{mn}, P^j] = \frac{1}{2}\epsilon^{imn}\big(\eta^{nj} P^m - \eta^{mj} P^n\big) \\ &= \frac{1}{2}\epsilon^{imn}\big(-\delta^{nj} P^m + \delta^{mj} P^n\big) = \frac{1}{2}\big(-\epsilon^{imj} P^m + \epsilon^{ijn} P^n\big) \\ &= \epsilon^{ijk} P^k \qquad (\text{rename dummies } m \to k,\ n \to k).\end{aligned}$$

For energy,

$$[J^i, H] = \frac{1}{2}\epsilon^{imn}[M^{mn}, P^0] = \frac{1}{2}\epsilon^{imn}\big(\eta^{n0} P^m - \eta^{m0} P^n\big) = 0.$$

5) $\underline{[K^i, H] = P^i}$ **and** $\underline{[K^i, P^j] = \delta^{ij} H}$**:**

$$[K^i, H] = [M^{i0}, P^0] = \eta^{00} P^i - \eta^{i0} P^0 = P^i.$$

$$[K^i, P^j] = [M^{i0}, P^j] = \eta^{0j} P^i - \eta^{ij} P^0 = 0 - (-\delta^{ij}) P^0 = \delta^{ij} H.$$

6) $\underline{[P^i, P^j] = 0}$ **and** $\underline{[P^i, H] = 0}$**:** These follow from $[P^\mu, P^\nu] = 0$.

Summary:

$$\boxed{\begin{array}{lll} [J^i, J^j] = \epsilon^{ijk} J^k, & [J^i, K^j] = \epsilon^{ijk} K^k, & [K^i, K^j] = -\epsilon^{ijk} J^k, \\ [J^i, P^j] = \epsilon^{ijk} P^k, & [J^i, H] = 0, & [K^i, H] = +P^i, \\ [K^i, P^j] = +\delta^{ij} H, & [P^i, P^j] = 0, & [P^i, H] = 0\ . \end{array}}$$

Note: The above results can also be derived using the following identity:

$$\epsilon^{ijk}\epsilon^{lmn} = \det \begin{pmatrix} \delta^{il} & \delta^{im} & \delta^{in} \\ \delta^{jl} & \delta^{jm} & \delta^{jn} \\ \delta^{kl} & \delta^{km} & \delta^{kn} \end{pmatrix} \tag{12}$$

Above I have solved without using it but anybody who has used the above identity also gets full marks!

(b) Physical meaning

- $[J^i, J^j] = \epsilon^{ijk} J^k$: spatial rotations don't commute and they form the Lie algebra $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ (check appendix D.1 and D.2 of the typed notes).
- $[J^i, K^j] = \epsilon^{ijk} K^k$: boosts transform as 3-vectors under rotations just like momentum (see below).
- $[K^i, K^j] = -\epsilon^{ijk} J^k$: two non-collinear boosts compose to a rotation (called the *Thomas–Wigner rotation*).
- $[J^i, P^j] = \epsilon^{ijk} P^k$: momentum is a 3-vector under rotations just like boost (see above).
- $[J^i, H] = 0$: energy is a scalar under spatial rotations and angular momentum is conserved.
- $[K^i, H] = P^i$, $[K^i, P^j] = \delta^{ij} H$: boosts mix energy and momentum (Lorentz mixing).
- $[P^i, P^j] = 0$: successive spatial translations commute.
- $[P^i, H] = 0$: spatial and time translations commute in flat spacetime and momentum is conserved.

(c) Subalgebras of the Poincaré algebra

Closed under commutators:

- $\{J^i\}$: $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

- $\{J^i, K^i\}$: (proper orthochronous) Lorentz algebra $\mathfrak{so}^+(1,3)$.
- $\{P^\mu\} = \{H, P^i\}$: spacetime translations $\mathbb{R}^{1,3}$.
- $\{J^i, P^j\}$: Euclidean/Galilean algebra consisting of infinitesimal Galilean transformations.
- $\{J^i, H\}$: rotations + time translations.
- $\{J^i, P^j, H\}$: spatial Euclidean + time translations (still closed since $[J, H] = [P, H] = 0$).

Not closed:

- $\{K^i\}$, because $[K^i, K^j] \sim J^k$.
- $\{K^i, P^j\}$, because $[K^i, P^j] \sim H$ and $[K^i, K^j] \sim J^k$.
- $\{K^i, H\}$, because $[K^i, H] = P^i$.
- $\{K^i, P^j, H\}$ without J^i, because $[K, K] \sim J$.

Q5. Spin of complex scalar field

Setup: For spatial l, we have, $\partial^l = -\partial_l$. Free complex scalar:

$$\mathcal{L} = \partial_\mu \phi^\dagger \, \partial^\mu \phi - m^2 \phi^\dagger \phi, \qquad T^{\mu\nu} = \partial^\mu \phi^\dagger \, \partial^\nu \phi + \partial^\mu \phi \, \partial^\nu \phi^\dagger - \eta^{\mu\nu} \mathcal{L}.$$

Rotation generators

$$J_i = \frac{1}{2} \epsilon_{ijl} M^{jl} = \int d^3x \, \epsilon_{ijl} \, x^j \, T^{0l}(x), \qquad T^{0l} = \partial^0 \phi^\dagger \, \partial^l \phi + \partial^0 \phi \, \partial^l \phi^\dagger.$$

Mode expansions:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} \, e^{-ik\cdot x} + b^\dagger_{\vec{k}} \, e^{+ik\cdot x} \right),$$
$$\phi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \left(a^\dagger_{\vec{k}} \, e^{+ik\cdot x} + b_{\vec{k}} \, e^{-ik\cdot x} \right), \qquad k \cdot x = E_{\vec{k}} t - \vec{k} \cdot \vec{x},$$

with $[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$, $[b_{\vec{k}}, b^\dagger_{\vec{p}}] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})$.

Derivatives:

$$\partial_t e^{\mp i(Et-\vec{k}\cdot\vec{x})} = \mp iE\, e^{\mp i(\cdots)}, \qquad \partial^l e^{\mp i(Et-\vec{k}\cdot\vec{x})} = \pm ik^l e^{\mp i(\cdots)}.$$

Hence

$$\partial^0\phi = -i\int \frac{d^3k}{(2\pi)^3\sqrt{2E_{\vec{k}}}}\Big(E_{\vec{k}}a_{\vec{k}}e^{-ik\cdot x} - E_{\vec{k}}b^\dagger_{\vec{k}}e^{+ik\cdot x}\Big),$$
$$\partial^l\phi = +i\int \frac{d^3k}{(2\pi)^3\sqrt{2E_{\vec{k}}}}\Big(k^l a_{\vec{k}}e^{-ik\cdot x} - k^l b^\dagger_{\vec{k}}e^{+ik\cdot x}\Big),$$

and similarly exchanging $a \leftrightarrow a^\dagger$, $b \leftrightarrow b^\dagger$ for $\phi^\dagger$.

(a) Operator form of J_i

Insert the derivatives in T^{0l} and $J_i = \epsilon_{ijl}\int d^3x\, x^j T^{0l}$. After keeping only the $a^\dagger a$ and $b^\dagger b$ sectors (the mixed ab, $a^\dagger b^\dagger$ drop after normal ordering), one uses (see appendix C.3 of the typed notes for a derivation of the following delta identity)

$$\int d^3x\, x^j\, e^{\,i(\vec{q}-\vec{k})\cdot\vec{x}} = -\,i(2\pi)^3\,\frac{\partial}{\partial q_j}\,\delta^{(3)}(\vec{q}-\vec{k}), \qquad (\star)$$

and the same for $e^{-i(\vec{q}-\vec{k})\cdot\vec{x}}$ (the sign from the exponent is compensated by the sign in $\partial/\partial(-q_j)$). Thus, e.g. in the $a^\dagger a$ sector,

$$J_i \supset \epsilon_{ijl}\int \frac{d^3q\, d^3k}{(2\pi)^6}\,\frac{E_{\vec{q}}k^l}{\sqrt{2E_{\vec{q}}2E_{\vec{k}}}}\,a^\dagger_{\vec{q}}a_{\vec{k}}\left[-\,i(2\pi)^3\,\frac{\partial}{\partial q_j}\delta^{(3)}(\vec{q}-\vec{k})\right] + \cdots$$

Integrate by parts in $\vec{q}$ and then do the $\vec{q}$-integral:

$$\int d^3q\; f(\vec{q},\vec{k})\,\partial_{q_j}\delta^{(3)}(\vec{q}-\vec{k}) = -\int d^3q\;(\partial_{q_j}f)\,\delta^{(3)}(\vec{q}-\vec{k}) \;\Rightarrow\; \vec{q}\to\vec{k}.$$

The pieces where the derivative hits the smooth energy prefactor cancel between the two halves $\partial^0\phi^\dagger\,\partial^l\phi$ and $\partial^0\phi\,\partial^l\phi^\dagger$ (after normal ordering), leaving only the *orbital* action when the derivative hits the annihilation operator. Denoting $\partial^l_{(\vec{k})} \equiv \partial/\partial k_l$ and normal-ordering (hats for normal-ordered operators),

$$\boxed{J_i = \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}^\dagger_{\vec{k}}\left(-\,i\,\epsilon_{ijl}\,k^j\,\partial^l_{(\vec{k})}\right)\hat{a}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}}\left(-\,i\,\epsilon_{ijl}\,k^j\,\partial^l_{(\vec{k})}\right)\hat{b}_{\vec{k}}\right].}$$

Equivalently, in vector form,

$$\boxed{J_i = \int \frac{d^3k}{(2\pi)^3} \left[\hat{a}^\dagger_{\vec{k}} \left(-i\,(\vec{k} \times \nabla_{\vec{k}})_i\right) \hat{a}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}} \left(-i\,(\vec{k} \times \nabla_{\vec{k}})_i\right) \hat{b}_{\vec{k}}\right],}$$

which is precisely the stated result. We used the Fourier identity ($\star$), integration by parts, and normal ordering. The over all minus sign above can be noted from the following: i^2 gives a minus sign and Eq.($\star$) gives another. Finally, integration by parts to move derivatives away from the delta function gives another minus sign.

(b) Action on a one-particle creator

Take the a-sector (the b-sector commutes with $a^\dagger_{\vec{p}}$):

$$J^{(a)}_i = \int \frac{d^3k}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{k}} \Big(- i\,\epsilon_{ijl}\, k^j\, \partial^l_{(\vec{k})} \Big) \hat{a}_{\vec{k}}.$$

Then

$$\begin{aligned}
[J^{(a)}_i, a^\dagger_{\vec{p}}] &= \int \frac{d^3k}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{k}} \Big(- i\,\epsilon_{ijl}\, k^j\, \partial^l_{(\vec{k})} \Big) [\hat{a}_{\vec{k}}, a^\dagger_{\vec{p}}] \\
&= \int \frac{d^3k}{(2\pi)^3}\, \hat{a}^\dagger_{\vec{k}} \Big(- i\,\epsilon_{ijl}\, k^j\, \partial^l_{(\vec{k})} \Big) (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p}) \\
&= \int d^3k\; \hat{a}^\dagger_{\vec{k}}\, (-i\,\epsilon_{ijl}\, k^j)\, \partial_{k_l} \delta^{(3)}(\vec{k} - \vec{p}).
\end{aligned}$$

Integrate by parts in $\vec{k}$ (discard boundary terms):

$$[J^{(a)}_i, a^\dagger_{\vec{p}}] = - \int d^3k\; \partial_{k_l} \Big[\hat{a}^\dagger_{\vec{k}}\, (-i\,\epsilon_{ijl}\, k^j) \Big]\, \delta^{(3)}(\vec{k} - \vec{p}).$$

The derivative acts only on k^j (the label of $a^\dagger_{\vec{k}}$ is not an operator variable) [1], giving

$$\partial_{k_l} \big(\epsilon_{ijl}\, k^j \big) = \epsilon_{ijl}\, \delta^j{}_l,$$

and using $\partial_{k_l} \delta^{(3)}(\vec{k} - \vec{p}) = -\partial_{p_l} \delta^{(3)}(\vec{k} - \vec{p})$ to transfer the derivative to $\vec{p}$ before the k-integration, one arrives at the compact generator action

$$\boxed{[J_i,\, a^\dagger_{\vec{p}}] = a^\dagger_{\vec{p}} \Big(- i\,\epsilon_{ijl}\, p^j\, \partial^l_{(\vec{p})} \Big).}$$

[1] Think of a series like $\sum_n A_n\, f(n)$. Now acting the series with ∂_n will lead to $\sum_n A(n)\, f'(n)$. So, A_n is a symbol labelled by n for every n and hence the derivative doesn't act on it.

Therefore, for a one-particle state $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$,

$$J_i|\vec{p}\rangle\Big|_{\vec{p}=\vec{0}} = 0,$$

showing that the single-particle states of the complex scalar are *spin-0*.

Q6. Complex scalar 2-pt functions and Feynman propagator

Let

$$\Delta^+(x,x') \equiv \langle 0|\phi(x')\phi^\dagger(x)|0\rangle, \qquad \Delta^-(x,x') \equiv \langle 0|\phi^\dagger(x)\phi(x')|0\rangle,$$

and

$$\Delta_F(x,x') \equiv \langle 0|T\big(\phi(x')\phi^\dagger(x)\big)|0\rangle, \qquad \Box \equiv \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2.$$

(a) Klein–Gordon equation for the Wightman functions

Using the on-shell momentum representation:

$$\Delta^+(x,x') = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}}\, e^{-ip\cdot(x'-x)}, \qquad p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2},$$

one has

$$(\Box_x + m^2)\, e^{-ip\cdot(x'-x)} = (-p^2 + m^2)\, e^{-ip\cdot(x'-x)} = 0,$$

since $p^2 = m^2$ on shell. The same applies to Δ^-. Hence

$$\boxed{(\partial_\mu\partial^\mu + m^2)\, \Delta^\pm(x,x') = 0.}$$

(b) Time-ordered propagator in terms of Wightman functions

By definition of time ordering,

$$\boxed{\Delta_F(x,x') = \theta(t'-t)\, \Delta^+(x,x') + \theta(t-t')\, \Delta^-(x,x')}\,.$$

(c) Green's function equation for $i\Delta_F$

Act on the x-argument:

$$(\Box_x + m^2)\, i\Delta_F = i\Big[\Box_x\big(\theta(t'-t)\Delta^+\big) + \Box_x\big(\theta(t-t')\Delta^-\big)\Big].$$

Using (a), only derivatives hitting the step functions survive. Since θ depends only on t,

$$(\Box_x + m^2)\, i\Delta_F = i\Big\{(\partial_t^2\theta(t'-t))\Delta^+ + 2(\partial_t\theta(t'-t))\,\partial_t\Delta^+ + (\partial_t^2\theta(t-t'))\Delta^- + 2(\partial_t\theta(t-t'))\,\partial_t\Delta^-\Big\}.$$

Use the distributions

$$\partial_t\theta(t-t') = \delta(t-t'), \quad \partial_t\theta(t'-t) = -\delta(t-t'), \quad \partial_t^2\theta(t-t') = \delta'(t-t'), \quad \partial_t^2\theta(t'-t) = -\delta'(t-t'),$$

to obtain

$$(\Box_x + m^2)\, i\Delta_F = i\Big\{\delta'(t-t')(\Delta^- - \Delta^+) + 2\,\delta(t-t')\big(\partial_t\Delta^- - \partial_t\Delta^+\big)\Big\}.$$

Apply $g(t)\delta'(t-t') = -g'(t')\delta(t-t')$ (which is integration by parts and identity is appendix C.1 of typed notes) to the first term:

$$(\Box_x + m^2)\, i\Delta_F = i\,\delta(t-t')\left\{-\partial_t(\Delta^- - \Delta^+) + 2\big(\partial_t\Delta^- - \partial_t\Delta^+\big)\right\}_{t=t'}.$$

At equal times $t = t'$,

$$\Delta^+(x,x') = \Delta^-(x,x'),$$

and from the canonical equal-time commutator of a complex scalar $[\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, $\pi = \partial_t\phi^\dagger$ one has

$$\big(\partial_t\Delta^- - \partial_t\Delta^+\big)\Big|_{t=t'} = \langle 0|[\pi(x), \phi(x')]|0\rangle_{t=t'} = -\,i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}').$$

Therefore

$$(\Box_x + m^2)\, i\Delta_F(x,x') = i\,\delta(t-t')\big(-\,i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\big) = \delta^{(4)}(x'-x).$$

Submit via Google Form!

Deadline: 25/11/2025