Problem Sheet 1

Notations & Conventions

Greek indices: $\{\mu, \nu, \rho\sigma, \ldots\}$ will denote spacetime indices and Latin indices: $\{i, j, k, l, \ldots\}$ will denote spatial indices. We will work with mostly minus metric convention as in the lectures: $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$. We will set $\hbar = c = 1$, mostly.

Questions

1. Consider a vector field $A^\mu(x)$ which transforms under a generic (proper orthochronous) Lorentz transformation as: $A'^\mu(x) = \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x)$. Consider the infinitesimal version of this such that the finite transformation Λ can be written as $\Lambda(\omega_{\rho\sigma}) = \exp(\omega_{\rho\sigma}\, L^{\rho\sigma})$. Thus, under these we will have:

$$A'^\mu(x) = \Big[\exp\big(\omega_{\rho\sigma}\, L^{\rho\sigma}\big)\Big]^\mu{}_\nu\; A^\nu(x)$$

   a) Show that:

$$L^{\rho\sigma} = M^{\rho\sigma} + \partial^\rho x^\sigma - \partial^\sigma x^\rho \tag{1}$$

   where $M^{\rho\sigma}$ are the infinitesimal Lorentz generators acting on the coordinate vectors x^μ as follows: $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu = \Big[\exp\big(\omega_{\rho\sigma}\, M^{\rho\sigma}\big)\Big]^\mu{}_\nu\; x^\nu$.

   Note: *This is exercise 8 in the lecture notes.*

2. Consider the following Lagrangian:

$$\mathcal{L}_1 = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{2}$$

   where $(\partial_\mu A^\mu)^2 = (\eta^{\mu\nu}\partial_\mu A_\nu)\,(\eta^{\rho\sigma}\partial_\rho A_\sigma)$.

   a) Derive the EoMs for this Lagrangian. Hint: Use the identity: $\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu$.

   b) Show that they coincide with the EoMs derived from the following Lagrangian:

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. In other words, you would have shown that $\mathcal{L}_2 - \mathcal{L}_1$ is a total derivative term and hence they give the same action upto boundary terms which we can discard.

The above Lagrangians give the action for the electromagnetic field without sources. A_μ is the electromagnetic potential and $F_{\mu\nu}$ is its field strength called the *Faraday tensor*. Notice that there is no kinteic term $(\partial_t A_0)^2$ for A_0 in $\mathcal{L}_1$.

c) What does this mean? Hint: Compute the canonical momenta of A_0. What do you get?

Define the electric and magnetic fields as: $F^i_{\ 0} = E^i$ and $F^i_{\ j} = \epsilon^i_{\ jk} B^k$. Here ϵ^{ijk} is the completely anti-symmetric Levi-Civita symbol with the convention: $\epsilon^{123} = +1$.

d) Check that you get the source-free versions of the Gauss's law for electrostatics and Ampére-Maxwell law:

$$\nabla \cdot \vec{E} = 0, \qquad \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0 \tag{4}$$

e) What happens to the other two Maxwell equations? What is a Lorentz covariant way to write them? Hint: Use the definition of $F_{\mu\nu}$ above, in terms of A^μ, and consider taking *cyclic-derivatives* of it.

Note: *This is exercise 10 in the lecture notes.*

3. Consider the action for the complex scalar field:

$$S[\Phi, \Phi^*] = \int d^4x \left((\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi \right). \tag{5}$$

a) Derive its equations of motion.

Note: *This is exercise 11 in the lecture notes.*

4. The conserved charges obtained for Poincaré transformations are

$$P^\mu = \int d^3x\, T^{0\mu}, \qquad M^{\mu\nu} = \int d^3x \left(x^\mu T^{0\nu} - x^\nu T^{0\mu} \right) = \int d^3x \left(J^0 \right)^{\mu\nu}, \qquad H = P^0. \tag{6}$$

Also define,

$$J^i = \tfrac{1}{2}\epsilon^{ijk}M_{jk}, \qquad K^i = M^{0i}. \tag{7}$$

a) Using above show that the Poincaré algebra becomes

$$[J^i, J^j] = \epsilon^{ijk}J^k, \quad [J^i, K^j] = \epsilon^{ijk}K^k, \quad [K^i, K^j] = -\epsilon^{ijk}J^k, \tag{8}$$

$$[J^i, P^j] = \epsilon^{ijk}P^k, \quad [J^i, H] = 0, \quad [K^i, H] = P^i, \tag{9}$$

$$[K^i, P^j] = \delta^{ij}H, \quad [P^i, P^j] = 0, \quad [P^i, H] = 0. \tag{10}$$

b) Explain what do each of the above commutators imply *physically*. For instance, $[P^i, P^j] = 0$, means that successive momenta commute.

c) Identify which of the above form sub-algebras of the Poincaré algebra. For instance, the following subset of Poincaré generators: $A = \{J^i, P^j\}$ form a sub-algebra but the subset: $B = \{K^i, P^j\}$ doesn't form a sub-algebra since commutator of K^i with itself gives J^i which takes us out of the subset B.

Hint: A sub-algebra of a Lie algebra is the set of all generators such that they map among themselves (or, to the zero vector) under the commutator structure. in other words, this sub-set is *closed* under the commutator structure.

Note: *This is exercise 16 in the lecture notes.*

5. Consider complex scalar field. Now consider the rotational generators as defined in question 4, that is, $J_i = \frac{1}{2}\epsilon_{ijl}M^{jl}$. Write this in terms of x^j and T^{0k} where $T^{\mu\nu}$ is the stress tensor.

a) Show that:

$$\begin{aligned}\hat{J}_i &= \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}^\dagger_{\vec{k}}\left(-i\epsilon_{ijl}k^j\partial^l_{(k)}\right)\hat{a}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}}\left(-i\epsilon_{ijl}k^j\partial^l_{(k)}\right)\hat{b}_{\vec{k}}\right], \\ &= \int \frac{d^3k}{(2\pi)^3}\left[\hat{a}^\dagger_{\vec{k}}\left(-i\vec{k}\times\nabla_k\right)_i\hat{a}_{\vec{k}} + \hat{b}^\dagger_{\vec{k}}\left(-\vec{k}\times\nabla_k\right)_i\hat{b}_{\vec{k}}\right], \end{aligned} \tag{11}$$

where $\partial^l_{(k)} \equiv \frac{\partial}{\partial k_l}$ that is derivative w.r.t. the l^{th} component of the 3-vector $\vec{k}$.

Hint: To get (11) use the following delta distribution identity:

$$\int d^3x\, x^j e^{i(\vec{q}-\vec{k})\cdot\vec{x}} = -\frac{i}{(2\pi)^3}\frac{\partial}{\partial q_j}\delta^{(3)}(\vec{q}-\vec{k}), \tag{12}$$

and then perform integration by parts (ignoring boundary terms) to move the derivative above away from the delta distribution. After this, do normal ordering too.

b) Next show that,

$$[\hat{J}_i, \hat{a}^\dagger_{\vec{p}}] = \hat{a}^\dagger_{\vec{p}}\left(-i\epsilon_{ijl}p^j\partial^l_{(p)}\right) \tag{13}$$

implying that $\hat{J}_i\left(\hat{a}^\dagger_{\vec{p}}|0\rangle\right)\Big|_{\vec{p}=0} = 0$ and hence the single-particle states are spin 0 states [1].

Note: *This is exercise 22 in the lecture notes.*

6. Consider the complex scalar field. Let us define:

$$\Delta^+(x,x') \equiv \langle 0|\phi(x')\phi^\dagger(x)|0\rangle, \qquad \Delta^-(x,x') \equiv \langle 0|\phi^\dagger(x)\phi(x')|0\rangle \tag{14}$$

These are called the *Wightman* functions.

a) Show that:

$$(\partial_\mu\partial^\mu + m^2)\Delta^\pm(x,x') = 0 \tag{15}$$

Hint: It will be better to work with the momentum space version of the Wightman functions!

Define the Feynman propagator as:

$$\Delta_F(x,x') \equiv \langle 0|T(\phi(x')\phi^\dagger(x))|0\rangle \tag{16}$$

where T denotes time-ordering.

[1] Do not put $\vec{p}=0$ before acting with $\hat{J}_i$ since that will be just like acting $\hat{J}_i$ on the vacuum itself which is Lorentz invariant as we will see in the next section when we discuss quantum symmetries.

b) Write $\Delta_F(x, x')$ in terms of $\Delta^+(x, x')$ and $\Delta^-(x, x')$.

c) Using (15) and result of b) from above, show that the the following holds:

$$(\partial_\mu \partial^\mu + m^2) i\Delta_F(x, x') = \delta^{(4)}(x' - x) \tag{17}$$

Hint: Use the identity: $\partial_t \theta(t - t') = \delta(t - t')$.

Submit via Google Form!

Deadline: 25/11/2025