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End Of Project Report

Title : *Classifying rational conformal field theories/vertex operator algebras using modular linear differential equations*

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Motivation and Scope of the Project

Two-dimensional Conformal Field Theories (2D CFTs) play a central role in both physics and mathematics. In physics, they describe critical phenomena in condensed matter systems and govern phase transitions. The worldsheet theory in String Theory is also a 2D CFT. A special class, known as Rational Conformal Field Theories (RCFTs), is of particular interest due to their finite spectrum of operators. In the mathematical literature, these objects are referred to as Vertex Operator Algebras (VOAs) and find applications in representation theory, affine Lie algebras, modular tensor categories, and lattice theory. A classification of RCFTs would therefore be of significant value to both communities.

This project aims to classify low-rank RCFTs using number-theoretic tools such as modular forms, automorphic forms, Rademacher series, and Modular Linear Differential Equations (MLDEs). The Mathur-Mukhi-Sen (MMS) approach consists of postulating a general holomorphic MLDE of a given order and selecting solutions whose series expansions have non-negative integer coefficients; these are then regarded as admissible characters. These characters are used to construct modular-invariant partition functions, and techniques from affine Lie algebras and modular tensor categories are employed to determine whether these characters indeed correspond to genuine RCFTs.

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Abstract

This project explores the use of modular linear differential equations (MLDEs) to classify rational conformal field theories (RCFTs). We begin by recalling the theory of modular forms, Eisenstein series, and the structure of the modular group $SL_2(\mathbb{Z})$. We then introduce the Serre−Ramanujan derivative operator, which allows us to construct modular differential equations. In the simplest non-trivial case ($n = 2$), the analysis of Frobenius-type solutions reveals arithmetic constraints on the central charge c and on the spectrum of conformal dimensions. These constraints naturally produce a discrete set of possible central charges, many of which correspond to known RCFTs. Thus, MLDEs provide a powerful bridge between number theory and theoretical physics, suggesting a systematic method for the classification of rational conformal field theories.

Keywords— modular forms, Eisenstein series, Serre–Ramanujan derivative, MLDE, RCFT.

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1.1 Introduction

Our report follows a clear progression: we begin by defining modular forms and their expansion in Fourier series (q-expansion), with the Eisenstein series as illustrative examples. We then explain the structure of the space of modular forms and its basis, the valence formula for determining zeros, and the central role of Klein's modular invariant $j(\tau)$. Next, we introduce the Serre–Ramanujan differential operator, which allows us to differentiate modular forms while preserving modularity, and we construct modular differential equations. In the particular case $n = 2$, we show that the solution develops as a Frobenius series whose coefficients $a_{i,j}$ have a physical interpretation: they count the states or particles of a system. By relating the initial exponent to the central charge c, we obtain arithmetic conditions that restrict the possible values of c. Finally, we identify a discrete set of central charges that correspond exactly to rational conformal field theories (RCFTs). Thus, the connection with physics is that these modular differential equations can serve to classify RCFTs.

1.2 Modular Forms on $SL_2(\mathbb{Z})$

We define the modular group as

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Definition 1.2.1. *A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half-plane $\mathbb{H}$ such that:*

1. *f is holomorphic on $\mathbb{H}$,*
2. *$f(\tau + 1) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$,*
3. *f is bounded as $\Im(\tau) \to \infty$.*

If in addition $f(\tau) \to 0$ as $\Im(\tau) \to \infty$, then f is called a cusp form.

Definition 1.2.2 (q-Expansion)**.** *Since a modular form is 1-periodic ($f(\tau + 1) = f(\tau)$), it admits a Fourier expansion colled q-expansion of f:*

$$f(\tau) = \sum_{n \geq 0} a_n e^{2\pi i n \tau} = \sum_{n \geq 0} a_n q^n, \quad q := e^{2\pi i \tau}.$$

The coefficients $a_n \in \mathbb{C}$ are called the Fourier coefficients of f.

Remark 1.2.1.

- *For $k = 0$, the only modular forms are constant functions.*
- *For $k < 0$ or k odd, the only modular form is the zero function.*

Definition 1.2.3 (Eisenstein series)**.** *For an even integer $k \geq 4$, the Eisenstein*

series of weight k is

$$G_k(\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^k}, \quad \tau \in \mathbb{H}.$$

Theorem 1.2.1. *For every even integer $k \geq 4$, the Eisenstein series $G_k(\tau)$ is a modular form of weight k for $SL_2(\mathbb{Z})$. Moreover,*

$$G_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

Definition 1.2.4 (Normalized Eisenstein Series)**.** *The normalized Eisenstein series is defined by*

$$E_k(\tau) := \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n, \quad q = e^{2\pi i\tau},$$

where B_k are the Bernoulli numbers.

For examples: $E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots$,
$E_6(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \cdots$.

Definition 1.2.5 (The Ramanujan Δ Function)**.** *The Ramanujan Δ Function is*

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots .$$

Note that $\Delta(\tau) \to 0$ as $\tau \to i\infty$, so Δ is indeed a cusp form.

Definition 1.2.6 (Eisenstein series of weight 2)**.** *For $\tau \in \mathbb{H}$, the Eisenstein series*

of weight 2 *is*

$$G_2(\tau) = 2\zeta(2) + \frac{2(2\pi i)^2}{(2-1)!}\sum_{n\geq 1}\sigma_1(n)q^n = \frac{\pi^2}{3} - 8\pi^2\sum_{n\geq 1}\sigma_1(n)q^n,$$

Definition 1.2.7 (Normalized Eisenstein series of weight 2)**.** *The normalized version is*

$$E_2(\tau) = \frac{G_2(\tau)}{2\zeta(2)} = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n.$$

Theorem 1.2.2. *For all* $\tau \in \mathbb{H}$*,*

$$E_2\big(-\tfrac{1}{\tau}\big) = \tau^2 E_2(\tau) - \frac{6i}{\pi}\tau.$$

Thus, unlike E_4 or E_6, the series E_2 is **not a modular form** since its transformation law contains the additional term $-\frac{6i}{\pi}\tau$. Instead, E_2 is called a *quasi-modular form*.

1.3 Generators E_4 and E_6

Let M_k be the set of modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$. It is a vector space over $\mathbb{C}$.

Theorem 1.3.1. *For any even integer* $k \geq 0$*, a basis of* M_k *is given by*

$$\big\{E_4^a E_6^b \ : \ a, b \geq 0,\ 4a + 6b = k\big\}.$$

Example 1.3.1. *For weight* 24*, we can write*

$$E_{24} \ = \ \frac{49\,679\,091}{236\,364\,091}\,E_6^4 + \frac{10\,285\,000}{236\,364\,091}\,E_4^6 + \frac{176\,400\,000}{236\,364\,091}\,E_4^3 E_6^2.$$

1.4 Valence Formula

Theorem 1.4.1 (Valence Formula)**.** *Let f a meromorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Then*

$$\mathrm{ord}_\infty(f) + \tfrac{1}{2}\,\mathrm{ord}_i(f) + \tfrac{1}{3}\,\mathrm{ord}_\rho(f) + \sum_{z \in \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})} \mathrm{ord}_z(f) = \frac{k}{12},$$

where $\rho = e^{2\pi i/3}$.

Example 1.4.1 (Application of the valence Formula)**.**

- *For E_4, the valence formula requires that the sum of the orders of the zeros is $\frac{k}{12} = \frac{4}{12} = \frac{1}{3}$. At ρ (which has a stabilizer of order 3), a simple zero counts as $\frac{1}{3}$, so the equality holds.*
- *For E_6, the weight 6 gives $\frac{k}{12} = \frac{6}{12} = \frac{1}{2}$. At the point i (stabilizer of order 2), a simple zero contributes $\frac{1}{2}$, which again satisfies the formula.*
- *Finally, Δ is a cusp form of weight 12, so it vanishes only at ∞, with a zero of order 1, which corresponds to $\frac{12}{12} = 1$.*

1.5 The Klein j-Function and Differential Operators

There exists a modular invariant function on the upper half-plane $\mathbb{H}$. The Klein j-function. It is defined by $j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}$, $\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}$. It satisfies $j(\gamma\tau) = j(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$.

Remark 1.5.1. *E_4 is holomorphic and $E_4(\omega) = 0$, Δ is holomorphic and $\Delta(i\infty) = 0$; $j(\tau)$ has weight 0 and diverges when $\tau \to i\infty$; Thus, j is not a modular form*

but a meromorphic modular function.

The ordinary derivative $f'(\tau)$ does not preserve modularity. To correct this flaw, we use $E_2(\tau)$ (which is itself not modular). The key fact is that the non-modular part of $\frac{1}{2\pi i} f'(\tau)$ is exactly canceled by the non-modular part of $-\frac{k}{12} E_2(\tau) f(\tau)$. Thus, by combining both terms, one obtains an operator that maps modular forms to modular forms.

Definition 1.5.1 (Serre-Ramanujan operator)**.** *For a modular form $f \in M_k$, define* $\partial_k(f)(z) = \frac{1}{2\pi i} \frac{df}{dz}(z) - \frac{k}{12} E_2(z) f(z)$.

Theorem 1.5.1. *The operator ∂_k maps $M_k \to M_{k+2}$, preserves cusp forms, and satisfies the Leibniz rule $\partial_{k+l}(g \cdot h) = \partial_k(g)h + \partial_l(h)g$.*

1.6 Modular Differential Equations

Definition 1.6.1. *Consider the linear differential operator $L = \sum_{j=0}^{n} \phi_j(\tau)\, D^j$, where we assume $\phi_j(\tau) \in M_{2n-2j}(\mathrm{SL}_2(\mathbb{Z}))$, $\quad (0 \le j \le n)$, so that each term $\phi_j D^j$ has total weight $2n$. Then the homogeneous modular differential equation reads $\phi_n(\tau) D^n + \phi_{n-1}(\tau) D^{n-1} + \cdots + \phi_1(\tau) D + \phi_0(\tau)\, x(\tau) \;=\; 0$.*

If $x(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, each term belongs to $M_{k+2n}(\mathrm{SL}_2(\mathbb{Z}))$ and the equality is taken in that space.

1.6.1 Case $n = 2$, $k = 0$

We consider the modular differential equation $\phi_2 D^2 x + \phi_1 Dx + \phi_0 x = 0$. Since $x \in M_0$, we must have $\phi_2 D^2 x \in M_4 \;\Rightarrow\; \phi_2 \in M_0 \;\Rightarrow\; \phi_2(\tau) = a$,
$\phi_1 Dx \in M_4 \;\Rightarrow\; \phi_1 \in M_2 \;\Rightarrow\; \phi_1(\tau) = 0$,
$\phi_0 x \in M_4 \;\Rightarrow\; \phi_0 \in M_4 \;\Rightarrow\; \phi_0(\tau) = -bE_4(\tau)$. Thus the equation becomes $\big(D^2 - \lambda E_4(\tau)\big) x(\tau) = 0, \qquad \lambda = \frac{b}{a}$.

1.6.2 Solution and connection to physics

Since this is a linear homogeneous equation of order 2, the solution space has dimension 2.

$$(D^2 - \lambda E_4(\tau))x(\tau) = 0. \tag{1}$$

The corresponding Frobenius solutions are

$$x_0(\tau) = q^{r_0} \sum_{N \geq 0} a_{N,0} q^N, \qquad x_1(\tau) = q^{r_1} \sum_{N \geq 0} a_{N,1} q^N. \tag{2}$$

More generally, we write

$$x(\tau) = q^r \sum_{N \geq 0} a_N q^N. \tag{3}$$

Then

$$D^2 x = \sum_{N \geq 0} (r+N)^2 a_N q^{r+N-1},$$

and the term involving E_4 is

$$-\lambda E_4(\tau)x(\tau) = -\lambda \sum_{N \geq 0} \left(\sum_{m=0}^{N} e_m^{(4)} a_{N-m} \right) q^{r+N},$$

where

$$E_2(\tau) = \sum_{m \geq 0} e_m^{(2)} q^m, \qquad E_4(\tau) = \sum_{m \geq 0} e_m^{(4)} q^m.$$

Identifying the coefficients of q^{r+N} gives, for all $N \geq 0$,

$$(r+N)^2 a_N - \frac{1}{6} \sum_{m=0}^{N} e_m^{(2)} (r+N-m) a_{N-m} - \lambda \sum_{m=0}^{N} e_m^{(4)} a_{N-m} = 0. \tag{4}$$

For $N = 0$, the relation (4) gives:

$$r^2 - \frac{1}{6}r - \lambda = 0.$$

Hence, the two roots r_0 and r_1 satisfy

$$r_0 r_1 = -\lambda, \qquad r_0 + r_1 = \tfrac{1}{6}.$$

For $N = 1$ we obtains

$$\left(\frac{1}{6} - \lambda\right) a_1 + (4r - 240\lambda)a_0 = 0.$$

In particular,

$$\left(\tfrac{1}{6} - \lambda\right) a_{0,1} + (4r_0 - 240\lambda)a_{0,0} = 0,$$
$$\left(\tfrac{1}{6} - \lambda\right) a_{1,1} + (4r_1 - 240\lambda)a_{1,0} = 0.$$

For $a_{0,0} = 1$, set

$$N = -120r_0 \quad \Longrightarrow \quad r_0 = -\frac{N}{120}.$$

After simplification, we obtain

$$N^2 + 22N + (N - 50)a_{0,1} = 0. \tag{5}$$

Connection with physics

In the equation

$$(D^2 - \lambda E_4)x = 0,$$

the variable $x(\tau)$ represents a physical quantity, and the coefficients $a_{i,j}$ of its Frobenius series count the particles in the system.

If we set

$$r_0 = -\frac{c}{24}, \qquad c \in \mathbb{Q} \quad \text{(central charge)},$$

then

$$5c = -120r_0 = N.$$

Since $a_{0,1}$ is an integer, Gauss theorem on polynomials with integer coefficients implies that N must also be an integer. Using (5)

$$a_{0,1} = \frac{N^2 + 22N}{50 - N},$$

we see that $N < 50$ in order for $a_{0,1}$ to remain positive and integral. Therefore,

$$c < 10, \qquad 5c \in \mathbb{N}.$$

Using equation (4) up to order $O(q^{10})$, we obtain

$$C = \left\{\tfrac{2}{5},\, 1,\, 2,\, \tfrac{19}{5},\, 4,\, \tfrac{26}{5},\, 6,\, 7,\, \tfrac{39}{5},\, 8\right\}.$$

These solutions correspond to Rational Conformal Field Theories (RCFTs). If a physical system is modularly invariant (e.g. the phase space of transitions of the states of water), its study amounts to analyzing a system defined on the torus. Since the torus is two-dimensional, the set C above constitutes the spectrum of an RCFT.

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