Application of Modular Forms in Physics

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Abstract

Modular forms [1] are holomorphic functions on the complex upper half-plane with prescribed transformation properties under the modular group. Their strong algebraic and differential structure makes them useful not only in number theory, but also in mathematical physics.

This report gives an introductory study of modular forms for the full modular group and their application to modular linear differential equations. We review the modular group, Eisenstein series, the structure of spaces of modular forms, the valence formula, the quasi-modular Eisenstein series E_2, and the Ramanujan-Serre derivative. We then introduce the Klein j-invariant and use modular derivatives to study first- and second-order modular differential equations, including equations with fixed and movable poles. By changing variables from the modular parameter τ to the j-invariant, certain second-order equations are reduced to ordinary differential equations with rational coefficients and, in the simplest cases, to Gauss hypergeometric equations.

Finally, we explain briefly how modular symmetry arises from the geometry of the torus and how solutions of modular differential equations appear as candidate characters in two-dimensional rational conformal field theory.

[1] A famous remark attributed to Martin Eichler states: "There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and modular forms."

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Chapter 1

Introduction

Modular forms occupy a remarkable position at the intersection of complex analysis, number theory, geometry and mathematical physics. Although their definition is analytic, their transformation properties under the modular group impose strong algebraic and arithmetic constraints. The same modular symmetry also appears naturally in physical systems defined on a torus, particularly in two-dimensional conformal field theory. This provides a setting in which ideas originating in number theory become useful tools for studying physical quantities.

One important manifestation of this connection is the theory of modular linear differential equations. Ordinary differentiation does not preserve modularity, so the study of differential equations compatible with modular transformations requires a modified differential operator. The Ramanujan-Serre derivative provides such an operator, allowing modular forms of a particular weight to be differentiated while retaining controlled modular behaviour. This makes it possible to construct differential equations whose coefficients are modularfunctions and whose solution spaces transform consistently under the modular group.

A second important object is the Klein j-invariant. Since every weight-zero modular function for the full modular group is a rational function of j, the j-invariant provides a natural coordinate for rewriting modular differential equations. In this variable, equations expressed originally in terms of Eisenstein series and the modular discriminant become ordinary differential equations with rational coefficients. For certain first- and second-order cases, this change of variable leads to explicit solutions, including solutions expressed in terms of hypergeometric functions.

The aim of this report is to give a self-contained introduction to these ideas and to explain how they connect with a simple physical application. Chapter 2 develops the basic theory of modular forms for the full modular group, including Eisenstein series, the structure of spaces of modular forms, the valence formula, E_2, and the Ramanujan and Serre derivatives. Chapter 3 introduces modular linear differential equations, studies first- and second-order examples with and without poles, and shows how the change from τ-space to j-space leads to hypergeometric

equations in the simplest cases. This chapter also briefly discusses the appearance of movable poles. Finally, Chapter 4 explains how modular symmetry arises from the geometry of the torus and how modular differential equations are related to characters of rational conformal field theories.

The overall purpose is therefore to illustrate how the analytic theory of modular forms leads naturally to a class of differential equations and how these equations provide a bridge between modular forms and two-dimensional conformal field theory.

Chapter 2

The Full Modular Group and Modular Forms

The theory of modular forms begins with the action of the modular group on the complex upper half-plane. The transformation properties imposed by this action lead to a distinguished class of holomorphic functions whose algebraic and differential structure will be important in later chapters. In particular, the finite-dimensionality of spaces of modular forms and the existence of weight-raising differential operators provide the framework in which modular linear differential equations arise.

2.1 The Full Modular Group

Let

$$\mathbf{H} = \{\tau \in \mathbf{C} : \mathrm{Im}(\tau) > 0\}$$

denote the complex upper half-plane. The *full modular group* is the special linear group of 2×2 matrices

$$\mathrm{SL}_2(\mathbf{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1 \right\}.$$

An element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

acts on $\mathbf{H}$ by the fractional linear transformation (Möbius transformation)

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

This action preserves the upper half-plane since

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} > 0.$$

The basic properties of this action are discussed in [4, pp. 1–3] and [6, pp. 11–13]. Of particular importance are the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which act respectively by

$$S\tau = -\frac{1}{\tau}, \qquad T\tau = \tau + 1.$$

Proposition 2.1.1. *The group* $\mathrm{SL}_2(\mathbf{Z})$ *is generated by* S *and* T.

This is the standard generating result for the modular group; see [4, p. 2 and pp. 7–8]. We shall use it without reproducing its proof. The matrices I and $-I$ induce the same transformation on $\mathbf{H}$. Consequently, the effective group of fractional linear transformations is

$$\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm I\}.$$

The importance of the generators S and T is that many transformation properties need only be checked under $\tau \longmapsto \tau+1$ and $\tau \longmapsto -\frac{1}{\tau}$. This considerably simplifies the definition and study of modular forms.

2.2 Modular Forms and Their Weights

The action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$ does not, in general, leave a modular form unchanged. Instead, a factor depending on the matrix and on τ appears. The exponent of this factor is called the *weight*.

Definition 2.2.1. Let $k \in \mathbf{Z}$. A meromorphic function $f : \mathbf{H} \to \mathbf{C}$ is said to be *weakly modular of weight* k for $\mathrm{SL}_2(\mathbf{Z})$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for every

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Since S and T generate the modular group, weak modularity is equivalent to the two transformation laws $f(\tau + 1) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau)$. The first

relation shows that f is periodic with period 1. Therefore, with

$$q = e^{2\pi i\tau},$$

a function holomorphic on $\mathbf{H}$ has a Fourier expansion near $i\infty$ of the form

$$f(\tau) = \sum_{n\in\mathbf{Z}} a_n q^n,$$

on the punctured unit disk $\mathscr{D}^*$ and $q : \mathbf{H} \to \mathscr{D}^*$. For modular forms, the holomorphy condition at infinity removes the negative powers of q. This leads to the standard definition [6, pp. 12–14].

Definition 2.2.2. A function $f : \mathbf{H} \to \mathbf{C}$ is a *modular form of weight* k for $\mathrm{SL}_2(\mathbf{Z})$ if

1. f is holomorphic on $\mathbf{H}$;
2. f satisfies
$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$
for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$;
3. f is holomorphic at the cusp $i\infty$ (bounded at $i\infty$), equivalently,
$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

The space of such forms is denoted by $M_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big)$.

If $a_0 = 0$, then f vanishes at $i\infty$ and is called a *cusp form*. The corresponding space is denoted $S_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big)$. Notice also that the matrix $-I$ belongs to $\mathrm{SL}_2(\mathbf{Z})$. Hence modularity gives $f(\tau) = (-1)^k f(\tau)$. Thus, for odd k,

$$M_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big) = \{0\}.$$

This elementary observation already places a strong restriction on the possible weights of nonzero modular forms [6, pp. 14–15].

2.2.1 Eisenstein Series

The fundamental examples of nonconstant modular forms are the Eisenstein series. For every even integer $k \geq 4$, define

$$G_k(\tau) = \sum_{\substack{(m,n)\in\mathbf{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}.$$

The series converges absolutely for $k \geq 4$, and G_k is a nonzero modular form of weight k for the full modular group [6, Proposition 2.6, pp. 15–16]. After normalization one obtains

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)},$$

where $\zeta(k) := \sum_{n=1}^{\infty} \frac{1}{n^k}$ for $\Re(k) > 1$. The most important Eisenstein series which will occur repeatedly are

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

and

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where

$$\sigma_r(n) = \sum_{d|n} d^r.$$

Their Fourier expansions are given in [6, pp. 19–20]. Another fundamental example is the discriminant modular form

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728}.$$

It is a cusp form of weight 12 whose Fourier expansion begins as:

$$\Delta(\tau) = q - 24q^2 + 252q^3 + \cdots.$$

Equivalently,

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \eta(\tau)^{24},$$

where

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \qquad q = e^{2\pi i \tau},$$

is the Dedekind eta function. It has weight $1/2$ with a multiplier system, meaning that

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(\gamma)(c\tau+d)^{1/2}\eta(\tau), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

where $\varepsilon(\gamma)$ is a twenty-fourth root of unity, with a consistent choice of square root. The exponent $1/2$ of $c\tau+d$ is the weight. Taking twenty-fourth powers removes the multiplier and gives

$$\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau).$$

Thus the weight of $\Delta = \eta^{24}$ is $24\cdot\frac{1}{2} = 12$. The first description appears in [6, p. 20], while the product representation is discussed in [6, pp. 98–100]. These examples also illustrate how weights behave under multiplication. If $f \in M_k\big(\mathrm{SL}_2(\mathbf{Z})\big)$ and $g \in M_\ell\big(\mathrm{SL}_2(\mathbf{Z})\big)$, then $fg \in M_{k+\ell}\big(\mathrm{SL}_2(\mathbf{Z})\big)$. Thus the weight behaves additively under products.

2.3 Vector Spaces of Modular Forms and Their Bases

For a fixed weight k, modular forms are closed under addition and scalar multiplication. Hence $M_k\big(\mathrm{SL}_2(\mathbf{Z})\big)$ is a complex vector space. Similarly, $S_k\big(\mathrm{SL}_2(\mathbf{Z})\big) \subseteq M_k\big(\mathrm{SL}_2(\mathbf{Z})\big)$ is a vector subspace. These elementary vector-space properties are discussed in [4, pp. 4–6]; finite-dimensionality is stated in [6, Theorem 3.1, p. 41]. The spaces are not merely finite-dimensional: for the full modular group their dimensions can be determined explicitly.

Theorem 2.3.1 (Dimension formula)**.** *Let k be a positive even integer. Then*

$$\dim M_k\big(\mathrm{SL}_2(\mathbf{Z})\big) = \begin{cases} \left\lfloor \dfrac{k}{12} \right\rfloor + 1, & k \not\equiv 2 \pmod{12}, \\ \left\lfloor \dfrac{k}{12} \right\rfloor, & k \equiv 2 \pmod{12}. \end{cases}$$

Moreover, $M_0\big(\mathrm{SL}_2(\mathbf{Z})\big) = \mathbf{C}$, while $M_k\big(\mathrm{SL}_2(\mathbf{Z})\big) = \{0\}$ for negative k and for odd k.

For the proof and the corresponding formula for cusp forms, see [6, Theorem 3.5, pp. 48–49]. A particularly useful consequence is that every modular form

for the full modular group is a polynomial in E_4 and E_6. More precisely,

$$M\Big(\mathrm{SL}_2(\mathbf{Z})\Big) = \bigoplus_{k\geq 0} M_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big) = \mathbf{C}[E_4, E_6].$$

This structure result is given in [6, Proposition 3.6, p. 49]; see also [4, Theorem 3.5.2, p. 88]. Consequently, for fixed weight k, a basis is obtained from all monomials

$$E_4^a E_6^b$$

satisfying

$$a, b \in \mathbf{Z}_{\geq 0}, \qquad 4a + 6b = k.$$

Thus

$$\mathscr{B}_k = \Big\{E_4^a E_6^b : a, b \in \mathbf{Z}_{\geq 0}, \quad 4a + 6b = k\Big\}$$

is a basis of $M_k(\mathrm{SL}_2(\mathbf{Z}))$. For example,

$$M_4 = \langle E_4 \rangle_{\mathbf{C}}\,, \qquad M_6 = \langle E_6 \rangle_{\mathbf{C}}\,,$$

$$M_8 = \left\langle E_4^2 \right\rangle_{\mathbf{C}}\,, \qquad M_{10} = \langle E_4 E_6 \rangle_{\mathbf{C}}\,,$$

and the space M_{12} is two-dimensional, with basis E_4^3 and E_6^2. Thus $M_{12} = \langle E_4^3, E_6^2 \rangle_{\mathbf{C}}$. The last example is especially important because

$$E_4^3 - E_6^2 = 1728\Delta.$$

The finite-dimensionality and explicit bases of these spaces will later make it possible to identify modular forms simply from their weights and a small amount of Fourier data.

2.4 The Fundamental Region and the Valence Formula

The modular transformation law relates the values of a modular form at different points of $\mathbf{H}$. It is therefore unnecessary to study the whole upper half-plane independently. One may instead restrict attention to a fundamental region for the action of the modular group.

Definition 2.4.1. A fundamental region for $\mathrm{SL}_2(\mathbf{Z})$ is a subset of $\mathbf{H}$ containing a representative of every $\mathrm{SL}_2(\mathbf{Z})$-orbit, with no two distinct interior points equivalent

under the modular group.

The standard fundamental region is

$$\mathscr{F} = \left\{ \tau \in \mathbf{H} : |\tau| \geq 1, \quad -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2} \right\}.$$

This is the standard region described in [6, pp. 25–26]. Its two vertical boundary components are identified by T, while the circular boundary is identified by S. Two special points in $\mathscr{F}$ are

$$i$$

and

$$\rho = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

They are the elliptic points of orders 2 and 3, respectively. Their nontrivial stabilizers are responsible for the fractional coefficients which appear in the valence formula. For a meromorphic function f, let $v_P(f)$ denote the order of vanishing of f at P, with a pole counted as a negative order. Let $v_\infty(f)$ be the smallest exponent of q occurring with nonzero coefficient in the Fourier expansion of f.

Theorem 2.4.1 (Valence formula)**.** *Let f be a nonzero weakly modular form of weight k for $\mathrm{SL}_2(\mathbf{Z})$. Then*

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{\substack{P \in \mathrm{SL}_2(\mathbf{Z}) \backslash \mathbf{H} \\ P \neq i, \rho}} v_P(f) = \frac{k}{12}.$$

This is Proposition 3.2 of Kilford [6, pp. 42–46]. Its proof is obtained by applying the argument principle to a contour constructed from the boundary of the standard fundamental region. We shall not reproduce the contour argument here. The valence formula gives strong information about the zeros of modular forms. For example, since E_4 has weight 4,

$$\frac{4}{12} = \frac{1}{3},$$

and its zero in the fundamental region is concentrated at the elliptic point ρ. Hence E_4 has a simple zero at ρ. Similarly, since E_6 has weight 6,

$$\frac{6}{12} = \frac{1}{2},$$

its zero is concentrated at the elliptic point i, where it has order one. For Δ, the

weight is 12 and its Fourier expansion begins with q. Thus

$$v_\infty(\Delta) = 1.$$

The valence formula then gives

$$1 = \frac{12}{12},$$

showing that Δ has no zeros in $\mathbf{H}$. Its only zero on the compactified modular curve is therefore the cusp at infinity. The valence formula is one of the principal reasons that modular forms are so rigid: the weight controls the total number of zeros, with the elliptic points counted according to their stabilizers. This rigidity also underlies the dimension and basis results of the preceding section.

2.5 The Eisenstein Series E_2

The Eisenstein series of weights $4, 6, 8, \ldots$ are genuine modular forms. Weight 2, however, behaves differently. The corresponding Eisenstein series is

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

It is holomorphic on $\mathbf{H}$ and at infinity, but it is not a modular form of weight 2. Kilford states this result in Proposition 2.9 [6, p. 21]. Indeed, under the transformation $S : \tau \mapsto -1/\tau$,

$$\tau^{-2} E_2\left(-\frac{1}{\tau}\right) = E_2(\tau) + \frac{12}{2\pi i \tau}.$$

Equivalently,

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{12\tau}{2\pi i}.$$

More generally, if

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

then the transformation law can be written as

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau + d).$$

The inhomogeneous or, non-modular term is discussed through the weight-2 Eisenstein series in [4, pp. 18–20]. Thus E_2 transforms *almost* as a modular form of weight 2, but with an additional term depending on c. This is consistent with the

dimension result

$$M_2\Big(\mathrm{SL}_2(\mathbf{Z})\Big) = \{0\}.$$

The function E_2 is therefore not modular; it is the basic example of a *quasi-modular form*. Kilford discusses this point in [6, pp. 37–38]. The failure of E_2 to be modular might initially appear inconvenient. In fact, its precise failure of modularity is exactly what is required to correct the failure of ordinary differentiation to preserve modularity.

2.6 Linear Operators on Modular Forms

It is useful to formulate modular transformation laws in terms of linear operators. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),$$

define the weight-k slash operator by

$$(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

For fixed γ, the map

$$f \longmapsto f|_k\gamma$$

is linear. Moreover,

$$(f|_k\gamma_1)|_k\gamma_2 = f|_k(\gamma_1\gamma_2).$$

The weight-k action is introduced in [6, p. 29]; its composition law is recorded in [6, p. 40, Exercise 19]. With this notation, the modularity condition simply becomes

$$f|_k\gamma = f$$

for every $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. There are also natural linear maps between modular-form spaces of different weights. For example, fixing $g \in M_r\Big(\mathrm{SL}_2(\mathbf{Z})\Big)$ gives a multiplication operator $\mathscr{M}_g : M_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big) \longrightarrow M_{k+r}\Big(\mathrm{SL}_2(\mathbf{Z})\Big)$, defined by $\mathscr{M}_g(f) = gf$. The fact that products add weights follows directly from the transformation law. For applications to differential equations, however, multiplication is not enough. We rather require a differential operator. The ordinary derivative is the natural candidate, but it does not preserve modularity. This leads to the Ramanujan-Serre derivative.

2.7 Ramanujan-Serre Derivative

Since modular forms have Fourier expansions in the variable

$$q = e^{2\pi i\tau},$$

the natural differential operator is

$$\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}.$$

Kilford calls θ the *Ramanujan operator* [6, p. 35]. If

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$\theta f = \sum_{n=0}^{\infty} n a_n q^n.$$

Thus θ acts particularly simply on Fourier expansions. However, if f has weight k, differentiating

$$f(\gamma\tau) = (c\tau + d)^k f(\tau)$$

shows that

$$f'(\gamma\tau) = (c\tau + d)^{k+2} f'(\tau) + kc(c\tau + d)^{k+1} f(\tau).$$

The second term prevents f' from transforming as a modular form of weight $k+2$. In particular,

$$\theta : M_k\big(\mathrm{SL}_2(\mathbf{Z})\big) \not\longrightarrow M_{k+2}\big(\mathrm{SL}_2(\mathbf{Z})\big)$$

in general. Kilford states this explicitly in Proposition 2.28 [6, p. 37]. The non-modular term can be cancelled using E_2.

Definition 2.7.1 (Ramanujan-Serre derivative)**.** For a modular form f of weight k, define

$$\mathscr{D}_k f = \theta f - \frac{k}{12} E_2 f.$$

Kilford uses the normalization $\partial_k = 12\theta - kE_2$, so that $\partial_k = 12\mathscr{D}_k$.

Theorem 2.7.1. *For every nonnegative integer k, the Serre derivative defines a linear weight-raising map*

$$\mathscr{D}_k : M_k\big(\mathrm{SL}_2(\mathbf{Z})\big) \longrightarrow M_{k+2}\big(\mathrm{SL}_2(\mathbf{Z})\big).$$

It also sends cusp forms to cusp forms.

This is Theorem 2.26 of [6, pp. 35–36], after dividing Kilford's operator ∂_k by 12. To see the cancellation explicitly, let $f \in M_k$ and

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

. Differentiating $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ gives

$$\frac{1}{2\pi i} f'(\gamma\tau) = (c\tau + d)^{k+2} \frac{f'(\tau)}{2\pi i} + \frac{kc}{2\pi i}(c\tau + d)^{k+1} f(\tau).$$

Using the transformation law

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) + \frac{12c}{2\pi i}(c\tau + d),$$

we therefore obtain

$$\begin{aligned} (\mathscr{D}_k f)(\gamma\tau) &= \frac{f'(\gamma\tau)}{2\pi i} - \frac{k}{12} E_2(\gamma\tau) f(\gamma\tau) \\ &= (c\tau + d)^{k+2} \frac{f'(\tau)}{2\pi i} + \frac{kc}{2\pi i}(c\tau + d)^{k+1} f(\tau) \\ &\quad - \frac{k}{12}(c\tau + d)^{k+2} E_2(\tau) f(\tau) - \frac{kc}{2\pi i}(c\tau + d)^{k+1} f(\tau) \\ &= (c\tau + d)^{k+2} (\mathscr{D}_k f)(\tau). \end{aligned}$$

Thus the two non-modular terms cancel. Holomorphy on the upper half-plane follows from that of f and E_2. Moreover, since $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$, the Fourier expansion of $\mathscr{D}_k f$ has no negative powers of q, and its constant term is $-\frac{k}{12}a_0$, where a_0 is the constant term of f. This proves holomorphy at the cusp and shows that cusp forms are sent to cusp forms.

The Ramanujan–Serre derivative also satisfies the expected product rule. If $f \in M_k$ and $g \in M_\ell$, then

$$\begin{aligned} \mathscr{D}_{k+\ell}(fg) &= \frac{f'g + fg'}{2\pi i} - \frac{k+\ell}{12} E_2 fg \\ &= \left(\frac{f'}{2\pi i} - \frac{k}{12} E_2 f\right) g + f\left(\frac{g'}{2\pi i} - \frac{\ell}{12} E_2 g\right) \\ &= (\mathscr{D}_k f) g + f(\mathscr{D}_\ell g). \end{aligned}$$

The corresponding identity for Kilford's normalization ∂_k is given in [6, p. 36].

2.7.1 Ramanujan Differential Identities

The interaction between θ, E_2, E_4, and E_6 gives three fundamental differential identities. First, Kilford's Exercise 17 shows that

$$E_2^2 - 12\theta E_2$$

is a modular form of weight 4 [6, p. 40]. Since $M_4\big(\mathrm{SL}_2(\mathbf{Z})\big) = \mathbf{C}E_4$ and both $E_2^2 - 12\theta E_2$ and E_4 have constant term 1, it follows that

$$E_2^2 - 12\theta E_2 = E_4.$$

Hence

$$\theta E_2 = \frac{E_2^2 - E_4}{12}.$$

Next, Kilford gives $\partial_4 E_4 = -4E_6$ on [6, p. 36]. Since $\partial_4 E_4 = 12\theta E_4 - 4E_2E_4$, we obtain

$$12\theta E_4 - 4E_2E_4 = -4E_6,$$

and therefore

$$\boxed{\theta E_4 = \frac{E_2E_4 - E_6}{3}}.$$

Similarly, $\partial_6 E_6 = -6E_8$. Since $E_8 = E_4^2$, we have $12\theta E_6 - 6E_2E_6 = -6E_4^2$, so

$$\boxed{\theta E_6 = \frac{E_2E_6 - E_4^2}{2}}.$$

Collecting the three identities gives the Ramanujan differential system

$$\begin{aligned}
\theta E_2 &= \frac{E_2^2 - E_4}{12}, \\
\theta E_4 &= \frac{E_2E_4 - E_6}{3}, \\
\theta E_6 &= \frac{E_2E_6 - E_4^2}{2}.
\end{aligned}$$

These equations show that differentiation does not take us outside the algebra generated by E_2, E_4, and E_6. In other words, the ring of quasi-modular forms is stable under θ; compare [6, pp. 37–38]. There is also an important identity involving the discriminant. Since $\Delta \in S_{12}\big(\mathrm{SL}_2(\mathbf{Z})\big)$ and the Serre derivative maps it to weight 14,

$$\mathscr{D}_{12}\Delta \in S_{14}\big(\mathrm{SL}_2(\mathbf{Z})\big).$$

But $S_{14}\Big(\mathrm{SL}_2(\mathbf{Z})\Big) = \{0\}$, and hence

$$\mathscr{D}_{12}\Delta = 0.$$

Therefore

$$\boxed{\theta\Delta = E_2\Delta}.$$

This relation is also indicated in [6, p. 39, Exercise 14]. Finally, because $\mathscr{D}_k$ raises the weight by 2, it can be iterated with the weight adjusted at each stage. Thus one may define

$$\mathscr{D}_k^{(n)} = \mathscr{D}_{k+2n-2} \circ \mathscr{D}_{k+2n-4} \circ \cdots \circ \mathscr{D}_{k+2} \circ \mathscr{D}_k,$$

so that

$$\mathscr{D}_k^{(n)} : M_k\Big(\mathrm{SL}_2(\mathbf{Z})\Big) \longrightarrow M_{k+2n}\Big(\mathrm{SL}_2(\mathbf{Z})\Big).$$

In particular, for an object of initial weight zero, $\mathscr{D}_0 = \theta$, but the second modular derivative is

$$\mathscr{D}_0^{(2)} = \mathscr{D}_2\mathscr{D}_0 = \left(\theta - \frac{1}{6}E_2\right)\theta,$$

rather than simply θ^2. This distinction between ordinary differentiation and modular differentiation will be essential when modular differential equations are introduced: the Ramanujan–Serre derivative allows differentiation while retaining controlled modular transformation properties.

Chapter 3

Modular Differential Equations

The differential structure discussed in the previous chapter allows one to construct differential equations whose coefficients are modular forms or modular functions. Such equations are called modular linear differential equations, abbreviated MLDEs. Their defining feature is that the differential operator is compatible with modular transformations. Consequently, the solution space is naturally preserved by the action of the modular group.

Before introducing these equations, it is useful to discuss the modular j-invariant. Besides being the basic modular function for the full modular group, the j-invariant provides a convenient change of variable that turns the modular coefficients of an MLDE into rational functions. This will make the singularities of the differential equation considerably more transparent.

3.1 The Modular j-Invariant

Recall that a modular form is required to be holomorphic on the upper half-plane and at the cusp. If the holomorphy condition is relaxed to meromorphy, one obtains the larger class of modular functions, else we have only constant modular-invariant modular forms.

Definition 3.1.1. A meromorphic function $f : \mathbf{H} \to \mathbf{C}$ is called a modular function of weight zero for $\mathrm{SL}_2(\mathbf{Z})$ if

$$f(\gamma\tau) = f(\tau)$$

for every $\gamma \in \mathrm{SL}_2(\mathbf{Z})$.

More generally, Kilford allows modular functions of arbitrary weight; see [6, Definition 5.1, pp. 94–95]. In this chapter we shall mainly need the weight-zero case.

The most important example is the Klein j-invariant,

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}.$$

Using

$$\Delta = \frac{E_4^3 - E_6^2}{1728},$$

this may also be written as

$$j(\tau) = 1728 \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}.$$

Its Fourier expansion begins

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots .$$

Thus j has a simple pole at the cusp $i\infty$. It is nevertheless holomorphic on $\mathbf{H}$, since Δ has no zeros there. These properties are discussed in [6, p. 95]; see also [4, pp. 6–7].

The special points introduced in the previous chapter are particularly simple in the j-coordinate. Since $E_4(\rho) = 0$ and $E_6(i) = 0$, one has

$$j(\rho) = 0, \qquad j(i) = 1728, \qquad j(i\infty) = \infty.$$

The three distinguished points of the modular fundamental region are therefore represented in the j-plane by 0, 1728, and ∞.

A fundamental reason for using j is the following structure theorem.

Theorem 3.1.1. *Every modular function of weight zero for the full modular group* $\mathrm{SL}_2(\mathbf{Z})$ *is a rational function of* j*. Conversely, every rational function of* j *is a modular function of weight zero.*

This is Theorem 5.3 of [6, pp. 95–96]. We shall use the result without reproducing its proof.

Thus, if f is a weight-zero modular function, then there exist polynomials P and Q such that

$$f(\tau) = \frac{P(j(\tau))}{Q(j(\tau))}.$$

This result is especially useful for differential equations. An equation whose coefficients originally involve E_4, E_6, and Δ can often be rewritten, after changing from τ to j, as an ordinary differential equation with rational coefficients.

Another useful identity follows immediately from the definition of j. For any constant p,

$$E_4^3 - p\Delta = \Delta(j - p).$$

Consequently, a zero of $E_4^3 - p\Delta$ corresponds precisely to the point $j = p$. This observation will later allow us to describe poles of an MLDE directly by their locations in the j-plane.

3.2 Modular Linear Differential Equations

Ordinary differentiation does not preserve modularity. For this reason the appropriate derivative in an MLDE is the Ramanujan–Serre derivative introduced in the previous chapter. We shall write

$$D_k = \theta - \frac{k}{12}E_2, \qquad \theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}.$$

When the weight is clear, we simply write D. Iteration always includes the appropriate change of weight. For example, when χ initially has weight zero,

$$D^2\chi = D_2 D_0 \chi = \left(\theta - \frac{1}{6}E_2\right)\theta\chi.$$

The general MLDE considered in [3, Sec. 2.2, pp. 4–6] has the form (See also [7] where it was first introduced)

$$\left(D^n + \sum_{s=1}^{n} \mu_{2s}\phi_{2s}(\tau)D^{n-s}\right)\chi(\tau) = 0.$$

Here ϕ_{2s} is a meromorphic modular form of weight $2s$, while μ_{2s} is a constant parameter. The weight assignment is necessary: since D raises modular weight by two, every term in the equation must have the same total modular weight.

The solutions need not themselves be scalar modular functions. In the applications to rational conformal field theory, several linearly independent solutions are collected into a vector and modular transformations mix these solutions among themselves. This is the origin of vector-valued modular forms in the MLDE approach [5, pp. 1–3]. (See also Gannon's review in [8])

3.2.1 The Wronskian and poles

Suppose an nth-order MLDE has linearly independent solutions $\chi_0, \ldots, \chi_{n-1}$. Their modular Wronskian is formed from the solutions and their successive modular derivatives. The coefficients of the MLDE can be expressed as ratios of

appropriate Wronskians. In particular, In particular, the modular analogue of Abel's identity gives

$$\mu_2\phi_2 = -D\log W,$$

relating the coefficient of the next-to-highest modular derivative to the logarithmic derivative of the Wronskian. See [3, Eq. (2.8), p. 6] and [5, Sec. 2, pp. 3–6].

This formula explains why poles naturally occur. If the Wronskian vanishes at a point, its logarithmic derivative develops a pole there. Thus zeros of the Wronskian become singularities of the coefficient functions of the MLDE.

The integer ℓ, called the *Wronskian index*, records the number of zeros of the Wronskian, with the appropriate fractional counting at the elliptic points. In particular, $\ell = 0$ corresponds to the nonsingular or monic case. When $\ell > 0$, poles can occur in the coefficient functions. For sufficiently large ℓ, these poles may occur at generic points of moduli space rather than only at the elliptic points [3, pp. 5–7].

Remark 3.2.1. A pole of an MLDE refers to a pole in one of its coefficient functions. The physical solutions need not have poles there. In the applications under consideration the solutions are required to remain regular and single-valued.

3.3 First-Order Modular Differential Equations

Before studying the two-character case, it is useful to see how the modular structure works in the simplest possible setting.

A first-order weight-zero MLDE has the form

$$(D_0 + \mu_2\phi_2)\,\chi = 0,$$

where ϕ_2 must have modular weight two. Since $D_0 = \theta$, this is

$$\theta\chi + \mu_2\phi_2\chi = 0.$$

3.3.1 The case with no poles

Suppose first that ϕ_2 is required to be holomorphic. For the full modular group there is no nonzero holomorphic modular form of weight two:

$$M_2\big(\mathrm{SL}_2(\mathbf{Z})\big) = \{0\}.$$

Hence the only first-order MLDE with holomorphic coefficients is $\theta\chi = 0$. Its solutions are simply $\chi(\tau) = C$, where C is constant.

This elementary example already illustrates why meromorphic coefficients are important. Nonconstant first-order solutions require a weight-two coefficient with poles.

3.3.2 A pole at $j = 0$: the case $\ell = 2$

The quotient E_6/E_4 has modular weight two and a simple pole where E_4 vanishes, namely at $\tau = \rho$. Consider

$$\theta\chi + \lambda\frac{E_6}{E_4}\chi = 0.$$

To solve this equation, differentiate the logarithm of $j = E_4^3/\Delta$. From the Ramanujan identities,

$$\theta \log j = 3\frac{\theta E_4}{E_4} - \frac{\theta\Delta}{\Delta} = \left(E_2 - \frac{E_6}{E_4}\right) - E_2 = -\frac{E_6}{E_4}.$$

The same identity is recorded as Eq. (2.28) in [3, p. 12]. Dividing the differential equation by χ, on a region where χ does not vanish, gives

$$\theta \log \chi = \lambda\theta \log j.$$

Integration therefore yields the local solution

$$\chi(\tau) = C\, j(\tau)^\lambda.$$

The exponent λ is not arbitrary if we require a holomorphic solution with a prescribed Wronskian index. Indeed, E_4 has a simple zero at ρ, while Δ does not vanish on the upper half-plane. Hence j has a zero of order three at ρ, so locally

$$j(\tau) = (\tau - \rho)^3 u(\tau), \qquad u(\rho) \neq 0.$$

Thus a nonzero solution Cj^λ extends holomorphically across ρ only if 3λ is a nonnegative integer. Its order of vanishing there is then 3λ.

For a first-order MLDE, the Wronskian is $W = \chi$. Recall that $\ell/6$ is the weighted number of zeros of W in the upper half-plane, modulo the modular group:

$$\frac{\ell}{6} = \frac{\operatorname{ord}_\rho W}{3} + \frac{\operatorname{ord}_i W}{2} + \sum_{p\neq\rho,i} \operatorname{ord}_p W,$$

where the sum runs over the remaining inequivalent points. In the present case, the only zero is at ρ, and therefore

$$\frac{\ell}{6} = \frac{3\lambda}{3} = \lambda.$$

Consequently, the case $\ell = 2$ fixes $\lambda = 1/3$, giving

$$\theta\chi + \frac{1}{3}\frac{E_6}{E_4}\chi = 0, \qquad \chi(\tau) = C\, j(\tau)^{1/3}.$$

The cube root can be chosen globally and holomorphically on the upper half-plane by writing

$$j^{1/3} = \frac{E_4}{\eta^8},$$

since $\eta^{24} = \Delta$ and η has no zeros there. This solution has a simple zero at ρ and transforms with a scalar multiplier, rather than being modular-invariant.

3.3.3 A pole at $j = 1728$: the case $\ell = 3$

There is an analogous construction using E_4^2/E_6, which has modular weight two and a simple pole at $\tau = i$. Since

$$j - 1728 = \frac{E_6^2}{\Delta},$$

the Ramanujan identities give

$$\theta \log(j - 1728) = 2\frac{\theta E_6}{E_6} - \frac{\theta\Delta}{\Delta} = \left(E_2 - \frac{E_4^2}{E_6}\right) - E_2 = -\frac{E_4^2}{E_6}.$$

Hence the equation

$$\theta\chi + \nu\frac{E_4^2}{E_6}\chi = 0$$

has the local solution

$$\chi(\tau) = C\,(j(\tau) - 1728)^{\nu}.$$

Again, holomorphy and the Wronskian index restrict the exponent. Since E_6 has a simple zero at i, the function $j - 1728$ has a zero of order two there. Locally,

$$j(\tau) - 1728 = (\tau - i)^2 v(\tau), \qquad v(i) \neq 0.$$

Thus holomorphy requires 2ν to be a nonnegative integer, and the solution has a

zero of order 2ν at i. There are no other zeros modulo the modular group, so

$$\frac{\ell}{6} = \frac{\operatorname{ord}_i \chi}{2} = \frac{2\nu}{2} = \nu.$$

Consequently, the case $\ell = 3$ fixes $\nu = 1/2$, and

$$\theta\chi + \frac{1}{2}\frac{E_4^2}{E_6}\chi = 0, \qquad \chi(\tau) = C\,(j(\tau) - 1728)^{1/2}.$$

A global holomorphic choice of the square root on the upper half-plane is

$$(j - 1728)^{1/2} = \frac{E_6}{\eta^{12}}.$$

This solution has a simple zero at i and also transforms with a scalar multiplier. However, holomorphy does not guarantee admissibility as an RCFT character. With the leading coefficient normalised to one, its expansion is

$$\frac{E_6}{\eta^{12}} = q^{-1/2}\,(1 - 492q + \cdots)\,,$$

so its negative Fourier coefficient rules out admissibility.

These two examples explain the appearance of factors j^δ and $(j - 1728)^\beta$ in the one-character discussion of [5, pp. 2–3]. For higher Wronskian indices, one considers expressions of the form

$$\chi(\tau) = j(\tau)^\delta (j(\tau) - 1728)^\beta P(j(\tau)),$$

where P is a nonzero polynomial. After absorbing integer powers of j and $j - 1728$ into P, the exponents may be chosen as

$$\delta \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}, \qquad \beta \in \left\{0, \frac{1}{2}\right\}.$$

If P has degree r, its roots contribute a total weighted zero count of r, including roots at the elliptic values. Therefore

$$\frac{\ell}{6} = \delta + \beta + r, \qquad \ell = 6\delta + 6\beta + 6r.$$

The polynomial factor thus accounts for additional zeros at higher Wronskian index. Further restrictions on the Fourier coefficients are required for these functions to be admissible characters.

These examples show that poles in the coefficients of an MLDE can coexist with

holomorphic solutions. Expressing the equation in terms of j makes the solutions explicit, while holomorphy and the Wronskian index constrain their exponents.

3.4 Second-Order Modular Differential Equations

We now turn to the case of two linearly independent solutions. The most general second-order MLDE has the form

$$\left(D^2 + \mu_2\phi_2 D + \mu_4\phi_4\right)\chi = 0,$$

where ϕ_2 and ϕ_4 have weights two and four respectively. This is the two-character specialization of the general MLDE discussed in [3, Sec. 2.3, pp. 9–13], (see also [7]).

We shall first study two basic cases: the equation with no poles and the first equation having a fixed pole.

3.4.1 The Pole-Free Case: $\ell = 0$

Suppose the coefficients are holomorphic. There is no nonzero modular form of weight two, so the coefficient of $D\chi$ must vanish. On the other hand, the space of weight-four modular forms is one-dimensional and is generated by E_4. Therefore the equation takes the form

$$\left(D^2 + \mu E_4\right)\chi = 0.$$

This is the Mathur–Mukhi–Sen equation, corresponding to $\ell = 0$; see [7].
To determine the relation between μ and the leading exponents of the solutions, use the Frobenius ansatz near the cusp:

$$\chi(\tau) = q^\alpha(1 + O(q)).$$

Since $E_2 = 1 + O(q)$ and $E_4 = 1 + O(q)$,

$$D_0\chi = \alpha q^\alpha + O(q^{\alpha+1}),$$

while

$$D^2\chi = D_2D_0\chi = \alpha\left(\alpha - \frac{1}{6}\right)q^\alpha + O(q^{\alpha+1}).$$

Substituting into the MLDE gives the indicial equation

$$\alpha\left(\alpha-\frac{1}{6}\right)+\mu=0.$$

If its roots are denoted by α_0 and α_1, then

$$\alpha_0+\alpha_1=\frac{1}{6},\qquad \mu=\alpha_0\alpha_1.$$

Hence the pole-free second-order equation can be written as

$$\left(D^2+\alpha_0\alpha_1 E_4\right)\chi=0.$$

When the indicial roots do not differ by an integer, the two independent solutions have Frobenius expansions

$$\chi_i(\tau)=q^{\alpha_i}\left(1+a_{i,1}q+a_{i,2}q^2+\cdots\right),\qquad i=0,1.$$

The differential equation then recursively determines the remaining Fourier coefficients once the leading normalization is chosen.

To obtain this recursion explicitly, let α be either indicial root and write

$$\chi(\tau)=q^{\alpha}\sum_{n=0}^{\infty}a_n q^n,\qquad a_0=1,\qquad \mu=\alpha\left(\frac{1}{6}-\alpha\right).$$

Using

$$E_2=1-24\sum_{m=1}^{\infty}\sigma_1(m)q^m,\qquad E_4=1+240\sum_{m=1}^{\infty}\sigma_3(m)q^m,$$

and $D^2=\theta^2-\frac{1}{6}E_2\theta$, comparison of the coefficient of $q^{\alpha+n}$ gives

$$\begin{aligned}0=&\left((\alpha+n)\left(\alpha+n-\frac{1}{6}\right)+\mu\right)a_n\\&+\sum_{m=1}^{n}\left(4\sigma_1(m)(\alpha+n-m)+240\mu\sigma_3(m)\right)a_{n-m}.\end{aligned}$$

The indicial equation simplifies the coefficient of a_n to

$$(\alpha+n)\left(\alpha+n-\frac{1}{6}\right)+\mu=n\left(n+2\alpha-\frac{1}{6}\right).$$

Thus, for $n \geq 1$,

$$a_n = -\frac{\displaystyle\sum_{m=1}^{n} \left(4\sigma_1(m)(\alpha + n - m) + 240\mu\sigma_3(m)\right) a_{n-m}}{n\left(n + 2\alpha - \frac{1}{6}\right)}.$$

Using Mathematica to implement this recursion with exact rational arithmetic, we obtain the first two Fourier coefficients:

$$a_1 = \frac{24\alpha(60\alpha - 11)}{12\alpha + 5},$$

and

$$a_2 = \frac{36\alpha\left(28800\alpha^3 - 8400\alpha^2 + 1016\alpha - 67\right)}{(12\alpha + 5)(12\alpha + 11)}.$$

Consequently,

$$\chi_i(\tau) = q^{\alpha_i}\left(1 + \frac{24\alpha_i(60\alpha_i - 11)}{12\alpha_i + 5}q + \frac{36\alpha_i\left(28800\alpha_i^3 - 8400\alpha_i^2 + 1016\alpha_i - 67\right)}{(12\alpha_i + 5)(12\alpha_i + 11)}q^2 + O(q^3)\right), \qquad i = 0, 1.$$

These formulas apply when the recursion denominators are nonzero. If the roots differ by an integer, the corresponding Frobenius step must be examined separately; logarithmic terms may be required. Coincident roots also require a separate construction of the second solution.

3.4.2 A Fixed Pole: $\ell = 2$

The next basic two-character equation is

$$\left(D^2 + \frac{E_6}{3E_4}D + \alpha_0\alpha_1 E_4\right)\chi = 0.$$

This is the $\ell = 2$ equation; see [3, Eq. (2.23), p. 11] and the detailed analysis in [5, Secs. 4–5, pp. 8–19].

The coefficient E_6/E_4 has a pole at $E_4 = 0$, hence at $\tau = \rho$. This is not a movable pole: its position is forced by the geometry of the modular group.

Again take $\chi \sim q^\alpha$. Since $E_6/E_4 \to 1$ at the cusp, the leading equation is

$$\alpha\left(\alpha - \frac{1}{6}\right) + \frac{\alpha}{3} + \alpha_0\alpha_1 = 0.$$

Equivalently,

$$\alpha\left(\alpha+\frac{1}{6}\right)+\alpha_0\alpha_1=0.$$

Thus the two roots satisfy

$$\alpha_0+\alpha_1=-\frac{1}{6}.$$

The presence of the pole has therefore changed the relation between the two leading exponents.

To determine the remaining Fourier coefficients, write

$$\chi(\tau)=q^\alpha\sum_{n=0}^{\infty}a_nq^n,\qquad a_0=1,\qquad \mu:=\alpha_0\alpha_1=-\alpha\left(\alpha+\frac{1}{6}\right).$$

Using $D^2=\theta^2-\frac{1}{6}E_2\theta$, the equation becomes

$$\left(\theta^2+B(q)\theta+\mu E_4\right)\chi=0,\qquad B(q):=-\frac{1}{6}E_2+\frac{E_6}{3E_4}.$$

The Eisenstein expansions give

$$\frac{E_6}{E_4}=1-744q+159768q^2+O(q^3),$$

and hence

$$B(q)=\frac{1}{6}-244q+53268q^2+O(q^3).$$

Write

$$B(q)=\sum_{m=0}^{\infty}b_mq^m,\qquad b_0=\frac{1}{6}.$$

Comparison of the coefficient of $q^{\alpha+n}$ gives

$$\begin{aligned}0=&\left((\alpha+n)\left(\alpha+n+\frac{1}{6}\right)+\mu\right)a_n\\&+\sum_{m=1}^{n}\left(b_m(\alpha+n-m)+240\mu\sigma_3(m)\right)a_{n-m}.\end{aligned}$$

Using the indicial equation, we obtain the recursion

$$a_n=-\frac{\displaystyle\sum_{m=1}^{n}\left(b_m(\alpha+n-m)+240\mu\sigma_3(m)\right)a_{n-m}}{n\left(n+2\alpha+\frac{1}{6}\right)},\qquad n\geq 1.$$

Using Mathematica to implement this recursion with exact rational arithmetic,

we obtain the first two Fourier coefficients:

$$a_1 = \frac{24\alpha(60\alpha + 71)}{12\alpha + 7},$$

and

$$a_2 = \frac{180\alpha\,(5760\alpha^3 + 14064\alpha^2 + 3592\alpha + 757)}{(12\alpha + 7)(12\alpha + 13)}.$$

Thus, when the indicial roots do not differ by an integer, the two independent solutions have the Frobenius expansions

$$\chi_i(\tau) = q^{\alpha_i}\left(1 + \frac{24\alpha_i(60\alpha_i + 71)}{12\alpha_i + 7}q + \frac{180\alpha_i\,(5760\alpha_i^3 + 14064\alpha_i^2 + 3592\alpha_i + 757)}{(12\alpha_i + 7)(12\alpha_i + 13)}q^2 + O(q^3)\right), \qquad i = 0, 1.$$

As in the pole-free case, resonant or coincident indicial roots require a separate Frobenius analysis. The normalization $a_0 = 1$ is used here for computation; the overall normalization of a non-vacuum character may subsequently be adjusted.

3.5 Changing from τ-Space to j-Space

The equations above are naturally written in the modular variable τ, but the j-invariant gives a more familiar ordinary differential equation. We now carry out this change of variable explicitly.

Let $\chi(\tau) = \chi(j(\tau))$. From the definition of j and the Ramanujan identities,

$$\theta j = -j\frac{E_6}{E_4}.$$

Consequently,

$$D_0\chi = \theta\chi = -j\frac{E_6}{E_4}\frac{d\chi}{dj}.$$

To compute the second modular derivative, recall that $D^2 = D_2D_0$. A direct calculation gives

$$D^2\chi = j^2\frac{E_6^2}{E_4^2}\frac{d^2\chi}{dj^2} + j\left(\frac{2}{3}\frac{E_6^2}{E_4^2} + \frac{1}{2}E_4\right)\frac{d\chi}{dj}.$$

The remaining modular forms can be eliminated using

$$\frac{E_6^2}{E_4^3} = 1 - \frac{1728}{j} = \frac{j - 1728}{j}.$$

This is the crucial step: after division by the leading coefficient, every remaining expression becomes rational in j.

3.5.1 The $\ell = 0$ Equation in j-Space

Substituting the preceding identities into

$$D^2\chi + \alpha_0\alpha_1 E_4\chi = 0$$

and dividing by the coefficient of the second derivative gives

$$\frac{d^2\chi}{dj^2} + \left(\frac{1}{2(j - 1728)} + \frac{2}{3j}\right)\frac{d\chi}{dj} + \frac{\alpha_0\alpha_1}{j(j - 1728)}\chi = 0.$$

This is precisely the $r = 0$, $\ell = 0$ specialization of the general j-space equation in [3, Eq. (3.25), pp. 17–19].

The singular points are now visible immediately:

$$j = 0, \qquad j = 1728, \qquad j = \infty.$$

They correspond respectively to $\tau = \rho$, $\tau = i$, and the cusp $i\infty$.

3.5.2 The $\ell = 2$ Equation in j-Space

The same calculation applied to

$$D^2\chi + \frac{E_6}{3E_4}D\chi + \alpha_0\alpha_1 E_4\chi = 0$$

gives

$$\frac{d^2\chi}{dj^2} + \left(\frac{1}{2(j - 1728)} + \frac{1}{3j}\right)\frac{d\chi}{dj} + \frac{\alpha_0\alpha_1}{j(j - 1728)}\chi = 0.$$

Again the coefficients are rational functions, and the only singular points are 0, 1728, and ∞.

This is the main advantage of the j-coordinate: complicated modular expressions in τ become rational coefficients with explicitly visible singularities.

3.6 Reduction to the Hypergeometric Equation

A second-order differential equation with the three regular singular points 0, 1, and ∞ is naturally compared with the Gauss hypergeometric equation

$$x(1-x)y'' + \Big(c - (a+b+1)x\Big)y' - ab\,y = 0.$$

The j-space equations above have singular points at 0, 1728, and ∞. Therefore the rescaling

$$x = \frac{j}{1728}$$

moves the finite singular points to 0 and 1.

3.6.1 Hypergeometric Form for $\ell = 0$

For the pole-free equation, substitution of $j = 1728x$ gives

$$x(1-x)\frac{d^2\chi}{dx^2} + \left(\frac{2}{3} - \frac{7}{6}x\right)\frac{d\chi}{dx} - \alpha_0\alpha_1\chi = 0.$$

Comparison with the Gauss equation gives

$$c = \frac{2}{3}, \qquad a+b = \frac{1}{6}, \qquad ab = \alpha_0\alpha_1.$$

But the indicial equation already gave $\alpha_0 + \alpha_1 = 1/6$. Hence one may choose

$$a = \alpha_0, \qquad b = \alpha_1.$$

Two independent solutions about $j = 0$ are therefore

$$\chi^{(1)}(j) = {}_2F_1\left(\alpha_0, \alpha_1; \frac{2}{3}; \frac{j}{1728}\right)$$

and

$$\chi^{(2)}(j) = \left(\frac{j}{1728}\right)^{1/3} {}_2F_1\left(\alpha_0 + \frac{1}{3}, \alpha_1 + \frac{1}{3}; \frac{4}{3}; \frac{j}{1728}\right).$$

These solutions are natural near $j = 0$. For character expansions, however, the cusp $q = 0$ is more important. Since $j \sim q^{-1}$, the variable

$$z = \frac{1728}{j}$$

satisfies $z \to 0$ as $q \to 0$.

Writing $\chi = z^\alpha f(z)$ and choosing $\alpha = \alpha_i$ gives the solution adapted to the cusp,

$$\chi_i(z) = C_i z^{\alpha_i} {}_2F_1\left(\alpha_i, \alpha_i + \frac{1}{3}; 1 + \alpha_i - \alpha_{1-i}; z\right), \qquad i = 0, 1.$$

Here and below, the two cusp solutions form a basis when the indicial roots do not differ by an integer. Resonant cases require a separate analysis, as in the Frobenius construction.

The constants C_i are fixed by the desired normalization of the Fourier expansions. Since $z \sim 1728q$, choosing $C_i = 1728^{-\alpha_i}$ gives leading coefficient one.

We now expand these solutions and compare them with the coefficients obtained from the MLDE recursion. From

$$j = q^{-1} + 744 + 196884q + O(q^2),$$

we obtain

$$u(q) := \frac{z}{1728q} = \left(1 + 744q + 196884q^2 + O(q^3)\right)^{-1} = 1 - 744q + 356652q^2 + O(q^3).$$

In particular,

$$z = 1728q - 1285632q^2 + O(q^3).$$

For either indicial root α, the normalized solution is therefore

$$\chi(\tau) = q^\alpha u(q)^\alpha {}_2F_1\left(\alpha, \alpha + \frac{1}{3}; 2\alpha + \frac{5}{6}; 1728qu(q)\right),$$

where we used $\alpha_0 + \alpha_1 = 1/6$ to simplify the third hypergeometric parameter.

The binomial expansion gives

$$\begin{aligned} u(q)^\alpha &= 1 - 744\alpha q + \left(356652\alpha + \frac{744^2}{2}\alpha(\alpha - 1)\right) q^2 + O(q^3) \\ &= 1 - 744\alpha q + \left(276768\alpha^2 + 79884\alpha\right) q^2 + O(q^3). \end{aligned}$$

On the other hand, the defining hypergeometric series gives

$${}_2F_1(\alpha, b; c; z) = 1 + \frac{\alpha b}{c} z + \frac{\alpha(\alpha+1)b(b+1)}{2c(c+1)} z^2 + O(z^3).$$

For convenience, write its first two coefficients as

$$A := \frac{\alpha b}{c}, \qquad B := \frac{\alpha(\alpha+1)b(b+1)}{2c(c+1)}.$$

Substituting the expansion of z gives

$$ {}_2F_1(\alpha, b; c; z) = 1 + 1728Aq + (-1285632A + 2985984B)\, q^2 + O(q^3). $$

Multiplying by $u(q)^\alpha$, we find

$$ \chi(\tau) = q^\alpha \left(1 + a_1 q + a_2 q^2 + O(q^3)\right), $$

where

$$ a_1 = -744\alpha + 1728A $$

and

$$ a_2 = 276768\alpha^2 + 79884\alpha - 1285632(\alpha + 1)A + 2985984B. $$

For the present equation, $b = \alpha + 1/3$ and $c = 2\alpha + 5/6$, so

$$ A = \frac{2\alpha(3\alpha + 1)}{12\alpha + 5}, \qquad B = \frac{2\alpha(\alpha + 1)(3\alpha + 1)(3\alpha + 4)}{(12\alpha + 5)(12\alpha + 11)}. $$

Consequently,

$$ \begin{aligned} a_1 &= -744\alpha + \frac{3456\alpha(3\alpha + 1)}{12\alpha + 5} \\ &= \frac{24\alpha(60\alpha - 11)}{12\alpha + 5}, \end{aligned} $$

and substitution into the expression for a_2 gives

$$ a_2 = \frac{36\alpha(28800\alpha^3 - 8400\alpha^2 + 1016\alpha - 67)}{(12\alpha + 5)(12\alpha + 11)}. $$

Thus the normalized cusp solutions have expansions

$$ \begin{aligned} \chi_i(\tau) = q^{\alpha_i} \Bigg(1 &+ \frac{24\alpha_i(60\alpha_i - 11)}{12\alpha_i + 5} q \\ &+ \frac{36\alpha_i(28800\alpha_i^3 - 8400\alpha_i^2 + 1016\alpha_i - 67)}{(12\alpha_i + 5)(12\alpha_i + 11)} q^2 + O(q^3) \Bigg), \qquad i = 0, 1. \end{aligned} $$

These coefficients agree exactly with those obtained earlier by implementing the $\ell = 0$ MLDE recursion in Mathematica.

3.6.2 Hypergeometric Form for $\ell = 2$

For the $\ell = 2$ equation, again set $x = j/1728$. The equation becomes

$$x(1-x)\frac{d^2\chi}{dx^2} + \left(\frac{1}{3} - \frac{5}{6}x\right)\frac{d\chi}{dx} - \alpha_0\alpha_1\chi = 0.$$

Comparison with the Gauss equation now gives

$$c = \frac{1}{3}, \qquad a + b = -\frac{1}{6}, \qquad ab = \alpha_0\alpha_1.$$

Since the indicial equation gave $\alpha_0 + \alpha_1 = -1/6$, we may again choose $a = \alpha_0$ and $b = \alpha_1$. Thus the solutions about $j = 0$ are

$$\chi^{(1)}(j) = {}_2F_1\left(\alpha_0, \alpha_1; \frac{1}{3}; \frac{j}{1728}\right)$$

and

$$\chi^{(2)}(j) = \left(\frac{j}{1728}\right)^{2/3} {}_2F_1\left(\alpha_0 + \frac{2}{3}, \alpha_1 + \frac{2}{3}; \frac{5}{3}; \frac{j}{1728}\right).$$

In terms of $z = 1728/j$, the basis adapted to the cusp, in the nonresonant case, is

$$\chi_i(z) = C_i z^{\alpha_i} {}_2F_1\left(\alpha_i, \alpha_i + \frac{2}{3}; 1 + \alpha_i - \alpha_{1-i}; z\right), \qquad i = 0, 1.$$

Again choosing $C_i = 1728^{-\alpha_i}$ and using $\alpha_0 + \alpha_1 = -1/6$, the normalized solution for either indicial root α becomes

$$\chi(\tau) = q^\alpha u(q)^\alpha {}_2F_1\left(\alpha, \alpha + \frac{2}{3}; 2\alpha + \frac{7}{6}; 1728qu(q)\right).$$

The expansions of $u(q)^\alpha$ and z are unchanged. Thus the preceding multiplication applies, now with $b = \alpha + 2/3$ and $c = 2\alpha + 7/6$. The first two hypergeometric coefficients are

$$A = \frac{2\alpha(3\alpha+2)}{12\alpha+7}, \qquad B = \frac{2\alpha(\alpha+1)(3\alpha+2)(3\alpha+5)}{(12\alpha+7)(12\alpha+13)}.$$

It follows that

$$\begin{aligned} a_1 &= -744\alpha + \frac{3456\alpha(3\alpha+2)}{12\alpha+7} \\ &= \frac{24\alpha(60\alpha+71)}{12\alpha+7}, \end{aligned}$$

while

$$a_2 = 276768\alpha^2 + 79884\alpha - 1285632(\alpha+1)A + 2985984B$$
$$= \frac{180\alpha(5760\alpha^3 + 14064\alpha^2 + 3592\alpha + 757)}{(12\alpha+7)(12\alpha+13)}.$$

Therefore,

$$\chi_i(\tau) = q^{\alpha_i}\left(1 + \frac{24\alpha_i(60\alpha_i+71)}{12\alpha_i+7}q \right.$$
$$\left. + \frac{180\alpha_i(5760\alpha_i^3 + 14064\alpha_i^2 + 3592\alpha_i + 757)}{(12\alpha_i+7)(12\alpha_i+13)}q^2 + O(q^3)\right), \qquad i = 0, 1.$$

These coefficients agree exactly with those obtained earlier by implementing the $\ell = 2$ MLDE recursion in Mathematica.

Thus both the $\ell = 0$ and $\ell = 2$ two-character equations reduce to Gauss hypergeometric equations. The difference between them is encoded in the local exponents and hence in the hypergeometric parameters. The direct expansions above verify that their cusp solutions reproduce the corresponding recursive Fourier expansions through q^2.

3.7 An Equation with a Movable Pole

The preceding $\ell = 2$ equation has a pole at a fixed special point of moduli space. A new phenomenon begins when the Wronskian is allowed to vanish at a generic point.

The first such two-character example occurs for $\ell = 6$. Let p denote the location of the Wronskian zero in the j-plane. In τ-space the equation takes the form

$$\left[D^2 + \frac{E_4^2 E_6}{E_4^3 - p\Delta}D + \alpha_0\alpha_1 E_4 \frac{E_4^3 - b\Delta}{E_4^3 - p\Delta}\right]\chi = 0.$$

Here p specifies the pole location while b is an accessory parameter. This is the $r = 1$ specialization of the $\ell = 6r$ family in [3, Sec. 2.3, pp. 9–11].

Using $E_4^3 - p\Delta = \Delta(j-p)$, the corresponding equation in j-space is

$$\frac{d^2\chi}{dj^2} + \left[\frac{1}{2(j-1728)} + \frac{2}{3j} - \frac{1}{j-p}\right]\frac{d\chi}{dj} + \frac{\alpha_0\alpha_1(j-b)}{j(j-1728)(j-p)}\chi = 0.$$

This is Eq. (3.28) of [3, pp. 20–21].

There are now four singular points:

$$0, \qquad 1728, \qquad p, \qquad \infty.$$

The extra point p is called a *movable pole* because its location is not fixed by the modular geometry. It is a parameter of the differential equation.

Near $j = p$, the indicial equation gives local exponents 0 and 2. Because their difference is an integer, a logarithmic solution can occur. Single-valuedness requires the logarithmic term to disappear. This imposes a relation between p and the accessory parameter b. Writing $A = \alpha_0\alpha_1$, the constraint is

$$A(p-b)^2 + \left(576 - \frac{5p}{6}\right)(p-b) + p(p-1728) = 0.$$

See Eqs. (3.32)–(3.33) of [3, pp. 20–21].

Unlike the $\ell = 0$ and $\ell = 2$ equations, this equation has an additional regular singular point at $j = p$. For $p \notin \{0, 1728\}$, its four regular singular points are $j = 0, 1728, p, \infty$. The change of variable $x = j/1728$ places them at $x = 0, 1, p/1728, \infty$, the standard configuration for Heun's equation. Thus the equation is of Heun type and does not generally reduce to the Gauss hypergeometric equation. Its solutions can be developed locally by the Frobenius method, while the no-logarithm condition at $j = p$ restricts the accessory parameters.

This additional singularity reflects the change in the possible pole structure of the modular coefficients as the Wronskian index increases:

- for $\ell = 0$, the modular coefficients are holomorphic;
- for nonzero $\ell < 6$, poles occur only at the special modular points $j = 0$ and $j = 1728$;
- beginning with $\ell = 6$ in the two-character case, poles may occur at generic locations $j = p$, whose positions enter as parameters of the equation.

3.8 Summary

The j-invariant provides the natural bridge between modular forms and ordinary differential equations. In τ-space, modular covariance is manifest but the coefficients are expressed through Eisenstein series and the discriminant. In j-space, the same equations become ordinary differential equations with rational coefficients.

For first-order equations, simple meromorphic weight-two coefficients lead directly to powers of j and $j - 1728$. For second-order equations, the pole-free $\ell = 0$

case and the fixed-pole $\ell = 2$ case both reduce to Gauss hypergeometric equations. Once a movable pole is introduced, an additional singular point and an accessory parameter appear.

These observations provide the differential-equation framework needed for the later application of MLDEs to conformal-field-theory characters.

Chapter 4

Modularity in Physics

The previous chapters introduced modular forms and modular differential equations from a mathematical point of view. We conclude by briefly explaining why modular symmetry arises naturally in two-dimensional conformal field theory. The aim is not to develop conformal field theory itself, but only to describe the geometric origin of modular invariance and its connection with the modular differential equations studied in the previous chapter.

The discussion follows the exposition of Das et.al in [1, 2]

4.1 From Finite Temperature to the Torus

One of the simplest ways in which a torus appears in physics is through a two-dimensional system at finite temperature. If the temperature is T, one writes $\beta = 1/T$. After passing to Euclidean time, the time coordinate becomes periodic,

$$t_E \sim t_E + \beta.$$

Thus the Euclidean time direction is a circle. If the spatial coordinate is also periodic, say (as in a crystal/lattice),

$$x \sim x + L,$$

then both directions are circles and the resulting Euclidean spacetime has the form

$$S^1 \times S^1 \simeq T^2.$$

This is the thermal torus described in [1, Secs. 2.4–2.5, pp. 5–6].

Introducing the complex coordinate $w = x + it_E$, the identifications for a rectangular torus are

$$w \sim w + L, \qquad w \sim w + i\beta.$$

After rescaling by L, they become

$$w \sim w + 1, \qquad w \sim w + \tau, \qquad \tau = i\frac{\beta}{L}.$$

More generally, a complex torus can be represented as

$$\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}), \qquad \tau \in \mathbf{H}.$$

The parameter τ therefore describes the shape of the torus.

The important point is that the choice of lattice basis is not unique. If one replaces the two generators by another basis of the same lattice, then the modular parameter transforms as

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Hence the modular group appears naturally as the group relating different descriptions of the same torus. This geometric origin of $\mathrm{SL}_2(\mathbf{Z})$ is explained in [1, Secs. 3.2–3.4, pp. 7–10].

In particular, the familiar generators

$$T : \tau \mapsto \tau + 1, \qquad S : \tau \mapsto -\frac{1}{\tau}$$

have a direct geometric interpretation in terms of changing the two cycles of the torus. Since these transformations do not change the underlying physical torus, physical quantities defined on it should respect this symmetry. This is the origin of modular invariance.

4.2 The Torus Partition Function and Characters

A central quantity associated with a two-dimensional conformal field theory on a torus is its partition function, written as

$$Z(\tau, \overline{\tau}).$$

Since two modularly related values of τ describe equivalent tori, the partition function is required to be modular invariant. Thus the geometry of the torus directly produces a modular constraint on the physical theory.

The same variable

$$q = e^{2\pi i\tau}$$

which appeared earlier in Fourier expansions of modular forms also has a physical interpretation. For the rectangular torus,

$$q = e^{-2\pi\beta/L}.$$

Therefore the limit $q \to 0$ corresponds to large β, or low temperature. In this sense, a q-expansion may also be viewed as a low-temperature expansion; see [2, Sec. 4.1, p. 6].

For a rational conformal field theory, the partition function can be expressed using finitely many holomorphic functions called *characters*. A character has a q-series of the form

$$\chi_i(\tau) = q^{h_i - c/24}\left(a_i(0) + a_i(1)q + a_i(2)q^2 + \cdots\right).$$

Here c is the central charge and h_i is the conformal weight associated with the corresponding state. The coefficients $a_i(n)$ count degeneracies of states at successive levels. This interpretation is given in [2, Sec. 4.2.2, pp. 7–8].

The partition function may then be written schematically as

$$Z(\tau, \overline{\tau}) = \sum_{i,j} M_{ij}\, \chi_i(\tau)\, \overline{\chi_j(\tau)},$$

where the coefficients M_{ij} are non-negative integers; see [2, Sec. 4.2.3, p. 8].

Although the complete partition function is modular invariant, the individual characters need not be invariant separately. Instead, modular transformations may mix the characters among themselves. Thus the finite collection of characters is preserved by the action of the modular group [2, Sec. 4.2.4, p. 8].

4.3 From Characters to Modular Differential Equations

We now recover the modular differential equations of the previous chapter.

Suppose a rational conformal field theory has finitely many linearly independent characters. These are holomorphic functions of τ, possess q-expansions, and transform among themselves under the modular group. It is therefore natural to look for a differential equation whose complete solution space has the same

modular behaviour.

Ordinary differentiation is not suitable because it does not preserve modularity. The modular derivative introduced earlier provides the required replacement. This leads naturally to a modular linear differential equation.

In the Mathur–Mukhi–Sen approach, the characters of a rational conformal field theory are studied as solutions of MLDEs subject to appropriate conditions on their q-expansions [2, Sec. 4.3, p. 8].

For example, if there are two independent characters, one is naturally led to a second-order MLDE. Writing a solution as

$$\chi_i(\tau) = q^{\alpha_i}\left(a_i(0) + a_i(1)q + a_i(2)q^2 + \cdots\right),$$

the leading exponent has the physical interpretation

$$\alpha_i = h_i - \frac{c}{24}.$$

For the vacuum character, $h_0 = 0$, and therefore

$$\alpha_0 = -\frac{c}{24}.$$

Thus the indicial exponents obtained when solving an MLDE are not merely parameters of a differential equation: they can encode physical quantities such as the central charge and conformal weights. Similarly, the Fourier coefficients of the solutions may be interpreted as degeneracies of physical states.

We therefore arrive at the basic connection underlying the application of modular forms considered in this report:

finite temp. in 2D → torus → modular symmetry → characters → modular differential equations.

The modular forms and modular derivatives introduced in the earlier chapters thus arise naturally in the study of physical theories defined on a torus. The geometry produces the modular symmetry, while MLDEs provide a mathematical tool for studying functions compatible with that symmetry and thereby finding new RCFTs

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