1 Problem Set - General Relativity - Date: 10/04/2018

1.1 Null Hypersurfaces and Event Horizons

Let us first define what we mean by a hypersurface.

Definition 1.1.1. Given a 4-dimensional spacetime $\mathcal{M}$, if we have a functional relation between its coordinates (x^α) then we generate a hypersurface. We define :-

$$\Sigma_S = \{x^\alpha \in \mathcal{M} \mid S(x^\alpha) = 0\} \tag{1}$$
$$\Sigma = \{x^\alpha \in \mathcal{M} \mid S(x^\alpha) = k(const.), k \in \mathbb{R}\} \tag{2}$$

Then, Σ_S is a 3-dimensional hypersurface and Σ is a collection of such hypersurfaces.

To understand the above let us consider the following example.

Example 1.1.1. Consider $\mathbb{R}^2$ (or, the Cartesian plane). The coordinates are, (x, y) and now let us consider a functional relation among them, say :-

$$x^2 + y^2 = a^2 \tag{3}$$

where $a(const.) \in \mathbb{R}$. Then, the above is a circle (centered at origin) which is a 1-dimensional hypersurface in the 2-dimensional Cartesian plane. In the above example, $S(x, y) = x^2 + y^2 = a^2$.

Now let us define what a normal vector field to a hypersurface Σ_S is :-

Definition 1.1.2. A vector field l_μ is called a normal vector field to Σ_S if it is of the form[1] :-

$$l_\mu = \partial_\mu S \tag{4}$$

Clearly, the above vector field is not normalized. Now let us characterize hypersurfaces based on their normal vector fields l_μ,

- if $l^\mu l_\mu < 0$ then, Σ_S is a timelike hypersurface
- if $l^\mu l_\mu > 0$ then, Σ_S is a spacelike hypersurface
- if $l^\mu l_\mu = 0$ then, Σ_S is a null hypersurface

For our discussion, we will only be concerned with null hypersurfaces. Now for null hypersurfaces $l^\mu l_\mu = 0$ and so, the normal vector field is tangent to the hypersurface itself, i.e.,

$$l^\mu = \frac{dx^\mu}{d\lambda} \tag{5}$$

for some null curve $x^\mu(\lambda)$ on Σ_S (where λ is the parameter on the curve). Now your first question is :-

Question 1.1.1. *Prove that $x^\mu(\lambda)$ is a null geodesic[2]. You might need to use the fact that, $\nabla_\alpha \partial_\beta S = \nabla_\beta \partial_\alpha S$ (try showing this too).*

We already know from class that an event horizon is a null hypersurface. Consider the Schwarzschild metric,

$$ds^2_{Sch} = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 \tag{6}$$

Now using the things given above,

Question 1.1.2. *Show that $r = 2M$ is indeed a null hypersurface and hence the event horizon for the Schwarzschild black hole. To solve this you may proceed as per the following steps :-*

[1] The reason being defining the normal this way is, since S is a constant on Σ_S its variation, i.e., $\partial_\mu S$ can only be in a direction orthogonal to Σ_S itself and hence is the normal.

[2] HINT: Try proving that l^μ satisfies the geodesic equation - $u^\beta \nabla_\beta u_\alpha \propto u_\alpha$ for a geodesic curve with tangent vector u. This is indeed a version of the geodesic equation you studied in class. To understand this think of the fact that a geodesic is a curve whose tangent vector is parallely propagated along itself.

- *Find Σ, i.e., the collection of hypersurfaces to which $r = 2M$ hypersurface belongs to.*
- *Compute the normal vector field l_μ for $r = 2M$ hypersurface.*
- *Compute the norm $l^\mu l_\mu$, on $r = 2M$ and from the value of the norm conclude that $r = 2M$ hypersurface is null.*

You must be knowing from class that a 2-dimensional sphere with radius a has the following metric :-

$$ds^2_{S^2} = a^2 d\theta^2 + a^2 sin^2\theta d\phi^2 \tag{7}$$

Question 1.1.3. *Prove that $r = 2M$ hypersurface has spherical topology, i.e., its metric is same as $eq^n(6)$ (upto a minus sign)*[3].

1.2 Killing Horizons and Surface Gravity

Let us define what we mean by a Killing horizon.

Definition 1.2.1. Let Σ_S be a null hypersurface. Let l_μ be its normal vector field. If l_μ happens to be a Killing vector field of the metric then Σ_S is said to be a Killing horizon.

Now consider the following question.

Question 1.2.1. *Prove that $r = 2M$ is a Killing horizon for the Schwarzschild metric. To solve this you may proceed as per the following steps.*

- *Find two most obvious Killing vector fields for ds^2_{Sch}.*
- *By computing the norm of both the above vector fields show that for $r > 2M$ one is spacelike and the other is timelike.*
- *Show that the Killing vector field which is timelike for $r > 2M$ becomes spacelike for $r < 2M$ and null for $r = 2M$ and hence the proof is done.*

You would have shown in $Ques^n$(1.1.1.) that,

$$l^\beta \nabla_\beta l_\alpha = \kappa l_\alpha \tag{8}$$

where κ is the proportionality constant. Now if l_μ is a Killing vector field then κ is known as the *surface gravity* of the null hypersurface which is already a Killing horizon.

Question 1.2.2. *Compute the surface gravity κ for the Schwarzschild metric and show that it is a constant, i.e., independent of the coordinates on the event horizon. To solve this you may proceed as per the following steps.*

- *Use $eq^n(7)$ to arrive at the equation :-*

$$\nabla_\alpha (l^\beta l_\beta) = 2\kappa l_\alpha \tag{9}$$

- *Now recall the expression for norm of the killing vector l_μ, which you used to prove that $r = 2M$ is a Killing horizon in $Ques^n$(1.2.1.). Use this expression in the LHS of $eq^n(8)$.*
- *For the RHS of $eq^n(8)$, compute the normal vector field for $r = 2M$. Now equating the LHS and RHS of $eq^n(8)$ and grouping similar terms obtain the value of the surface gravity κ.*

Rest of the material in this problem set is just for the sake of knowledge. From the above five questions we conclude that :-

- The event horizon is a null hypersurface.
- The event horizon has spherical topology. This fact is generally true for *stationary* 4-dimensional black holes.

[3]HINT: Observe $r = 2M$ at an instant of time, i.e., put $t = const.$ in ds^2_{Sch}.

- The event horizon is a Killing horizon. This fact is generally true for *stationary* 4-dimensional black holes and is known as the *Hawking Rigidity Theorem*.
- The surface gravity κ of the event horizon is constant throughout the horizon. This fact is a consequence of a general result known as the *Zeroth Law of Black Hole Mechanics*.

For interested readers, the constancy of the surface gravity κ across the horizon is the starting point for the field of Black Hole Thermodynamics and the temperature of a black hole is related to the surface gravity as :-

$$T_H = \frac{\kappa}{2\pi} \tag{10}$$

where T_H is the celebrated *Hawking Temperature*.