A derivation of the FLRW metric

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We provide a derivation of the FLRW metric from the symmetries of the spacetime which arises from the Cosmological Principle. We first review a bit of GR and then progress onto the complete derivation. We will focus on pedagogy rather than complete rigor but mostly we will give a basic meaning plus a rigorous touch to most of the things. We further assume that the vector fields we deal with in the discussion are all smooth.

Keywords: FLRW metric, symmetries of spacetime, Killing vectors, maximally symmetric space, constant spatial curvature

BASICS OF GR

Einstein's General Relativity (GR) till date is the most correct theory of gravity known to humankind. Recently, it has been shown by Collett et al.[1] that GR is true even at galactic length scales. So, let us quickly review it before proceeding on to derive the FLRW metric (which as we will see along the way is the correct metric for our universe).

Metric

A metric is a function of spacetime coordinates which determines distance between two nearby spacetime points. In more rigorous sense, a metric tensor (which is more important than the metric itself) is a symmetric and non-degenerate (0,2) tensor or a 2nd rank covariant tensor[1]. The presence of such a tensor allows to take an inner product of vectors in a vector space and hence with the help of metric tensor $g_{\mu\nu}$ we can define the metric to be (see [2, 3]),

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

which is exactly the distance between two spacetime points and since its a scalar by definition it must be preserved under general coordinate transformations or in a more rigorous sense under diffeomorphisms which preserve the metric (i.e. isometries). With this in hand we move onto analysis and hence differentiation.

Affine connection

As the name suggests it is indeed a connection between two vector spaces. So if we want to calculate the rate change of a vector along a path which moves from one vector space to another we can make use of this affine connection. So what is really going on is this, in usual flat spacetime or in $\mathbb{R}^n$ for $n \in \mathbb{N}$, at a point in the spacetime the vector space that exists is isomorphic (a stylish word for equivalent) to every other vector spaces existing at other points in the spacetime. Hence, when we move the vector along any curve whatsoever and calculate its rate change we do not care and simply differentiate the vector w.r.t. the curve parameter (which is time t mostly in physics) and we end up with a vector which still lies in the same vector space (as all vector space are isomorphic). But this is not generally true in GR where the spacetime is not generally flat.

This can be checked by noting that ordinary derivative of a vector is not a tensor under general coordinate transformations. So, to get a tensor out of differentiation you subtract the non-linear part from the above derivative and end up with something known as a covariant derivative which is indeed a tensor. The part you subtract is indeed the affine connection (see [2]) which is a (1,2) tensor denoted by $\Gamma^\phi_{\alpha\beta}$. For a contravariant vector, it reads as follows (see [3, 4]),

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha}A^\alpha. \tag{2}$$

One can find a similar expression for tensors of other ranks (see [3, 4]). As of now, we only know that $\Gamma^\phi_{\alpha\beta}$ is the non-linear term appearing in the expression for ordinary derivative of a vector. So it is given by,

$$\Gamma^\phi_{\alpha\beta} = \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\phi}{\partial x'^\mu}. \tag{3}$$

Parallel Transport

The notion of parallel transport is central to the understanding of various things in GR. First of all let us pick a curve γ with u^α as its tangent vector. Now say we want to parallel transport a vector a^μ along γ (a transport such that the vector remains parallel to itself after transportation). Finally, we would like to compute its rate change along γ. So, in order to compute its rate

[1] You can argue these properties of a metric tensor as follows. Since a metric tensor is our only hope to define inner product between two vectors in a particular vector space it better be symmetric else its contraction with the vectors would vanish under a symmetric exchange of the vector indices. Now non-degenerate means the associated 4×4 matrix would be non-singular and we require this to define an inverse of the metric tensor itself.

change along γ we need to take its covariant derivative and then project it along the curve γ, i.e., by taking its inner product with u^α. Now we would like this quantity to be zero as recall in Euclidean geometry, heuristically speaking parallel transport of a vector along a curve captures the notion of such a transport which preserves everything regarding the vector[2]. Hence, for parallel transport we must have,

$$u^\mu \nabla_\mu a^\alpha = 0. \tag{4}$$

Assumptions of GR

The following are the assumptions of GR,

- Covariant derivative of the metric tensor is zero.

$$\nabla_\alpha g_{\mu\nu} = 0 \tag{5}$$

- The affine connection is symmetric in its lower two indices.

$$\Gamma^\phi_{\alpha\beta} = \Gamma^\phi_{\beta\alpha} \tag{6}$$

Let us try and understand what these assumptions mean. The first assumption has two different interpretations. Firstly, it means that when you parallel transport a vector along a curve, then its norm remains same after it has been parallely transported along a closed curve. Secondly, it is a way to pick up a unique covariant derivative operator. We choose our derivative operator to be such which when acts on the metric gives zero.

The second assumption (technically called the torsion free assumption) is a bit difficult and it simply means that,

$$\nabla_\alpha \nabla_\beta f = \nabla_\beta \nabla_\alpha f$$

where f is a scalar function of the spacetime coordinates. One can easily check that the above gives rise to the second assumption.

Using the first assumption, one can show that (see [2]),

$$\Gamma^\phi_{\alpha\beta} = \frac{1}{2} g^{\phi\lambda} \left[\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right]. \tag{7}$$

[2] Imagine drawing a square on a paper and having a vector (pointing up) on its top right edge. Now transport the vector along the square in the anti-clockwise direction. When you return to the original position you can see that the vector obtained after parallel transport along the square is exactly same as the original vector in both direction and magnitude. But if your spacetime has curvature this won't happen and your final vector might be rotated w.r.t. your original vector.

Now demanding $eq^n(2)$ to behave like a tensor under coordinate transformations, we arrive at the transformation law for the affine coonection as follows,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^\phi} \frac{\partial x^\alpha}{\partial x'^{\mu}} \frac{\partial x^\beta}{\partial x'^{\nu}} \Gamma^\phi_{\alpha\beta} + \frac{\partial^2 x^\rho}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^\rho}. \tag{8}$$

One can also derive the above from $eq^n(7)$.

Geodesics

Geodesics are the shortest distanced curve on a spacetime the analogue of straight lines in flat spacetime. Also, they are defined as the straightest possible curve or technically a curve whose tangent vector is parallely propagated along itself and this means the geodesic equation must look like (from $eq^n(4)$),

$$u^\mu \nabla_\mu u^\nu = 0 \tag{9}$$

where u^μ is tangent to a curve γ which is a geodesic. The above equation can also be derived from extremising the length between neighbouring points (see [3]), i.e., by extremising the following integral,

$$I = \int_1^2 d\lambda \left[g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right]^{1/2} \tag{10}$$

where the integrand is the distance between points 1 and 2 and λ is the curve parameter.

Riemann Normal coordinates

We would now discuss a very peculiar coordinate system where most of the calculations in GR simplify a lot. Let us define this as,

$$x'^{\mu} = x^\mu + \frac{1}{2} Q^\mu_{\alpha\beta} x^\alpha x^\beta, \tag{11}$$

where $Q^\mu_{\alpha\beta} = Q^\mu_{\beta\alpha}$ are constants to be chosen as per convenience. Now if we plug $eq^n(11)$ into $eq^n(8)$ and say that we were at the origin such that $x^\mu = 0$, then a straightforward calculation shows that,

$$\Gamma'^{\mu}_{\alpha\beta} = 0, \tag{12}$$

if we choose $Q^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta}\Big|_{x^\mu=0}$. Thus, there exists a coordinate system where the affine connection vanishes at a point and this point being arbitrary can be anywhere. So, in a small enough neighbourhood of a point we can make an arbitrary spacetime look locally flat. This thing motivated Einstein to discover the *Equivalence Principle*. Also, note that in this coordinate system,

$$\frac{\partial^2 x^\phi}{\partial x'^{\alpha} \partial x'^{\beta}} \frac{\partial x'^{\mu}}{\partial x^\phi} = 0,$$

implying that,

$$\frac{\partial^2 x^\phi}{\partial x'^\alpha \partial x'^\beta} = 0,$$

which in turn implies (see [4]),

$$g'_{\mu\nu} = \eta_{\mu\nu}, \tag{13}$$

in this coordinate system around that point.

The Riemann tensor

We define the most important Riemann tensor as the term which prevents from mixed covariant derivatives to commute while acting on a vector,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\lambda = -R^\rho_{\lambda\mu\nu} A_\rho, \tag{14}$$

where $R^\rho_{\lambda\mu\nu}$ is the Riemann tensor which basically describes the curvature of the spacetime. We refer to [2, 3] to understand why Riemann tensor captures the notion of curvature in any spacetime and its all components are zero for a flat spacetime. Moving onto the symmetries of the Riemann tensor we have (you can compute these symmetries below by working out the Riemann tensor in Riemann normal coordinates),

$$R_{\rho\lambda\mu\nu} = -R_{\rho\lambda\nu\mu}, \tag{15}$$
$$R_{\rho\lambda\mu\nu} = -R_{\lambda\rho\mu\nu}, \tag{16}$$
$$R_{\rho\lambda\mu\nu} = R_{\mu\nu\rho\lambda}, \tag{17}$$
$$R_{\rho\lambda\mu\nu} + R_{\rho\nu\lambda\mu} + R_{\mu\nu\lambda} = 0. \tag{18}$$

We also have two very important identities related to its derivative,

$$R_{\rho\lambda\mu\nu;\sigma} + R_{\rho\lambda\sigma\mu;\nu} + R_{\rho\lambda\nu\sigma;\mu} = 0, \tag{19}$$
$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)^\nu_{;} = 0. \tag{20}$$

where $R_{\lambda\nu} = g^{\rho\mu} R_{\rho\lambda\mu\nu}$ is called the Ricci tensor (which is symmetric) and $R = g^{\lambda\nu} R_{\lambda\nu}$ is called the Ricci scalar. $Eq^n(19)$ is called the Bianchi identity and $eq^n(20)$ is called the contracted Bianchi identity.

Einstein Field Equations

Einstein postulated that matter curves spacetime and spacetime through its curvature dictates matter how to move. We already have a tensor describing some form of matter known as the stress tensor $T_{\mu\nu}$ (found in fluid mechanics) which has energy density and momentum density as its elements (nothing surprising as energy is mass). In the curvature sector we have a 2nd rank tensor $R_{\mu\nu}$. So, Einstein first postulated that,

$$R_{\mu\nu} = \kappa T_{\mu\nu}, \tag{21}$$

and now from the conservation equation,

$$T_{\mu\nu;}{}^{\nu} = 0, \tag{22}$$

we have $R_{\mu\nu;}{}^{\nu} = 0$ which is generally not true as in an arbitrary spacetime $R_{\mu\nu}$ has 10 independent components (due to its symmetry) but $R_{\mu\nu;}{}^{\nu} = 0$ further restricts the independent components of $R_{\mu\nu}$. So, Einstein took a brilliant guess and using Bianchi identity guessed the equation of motion as,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{23}$$

which is the correct EOM. One can determine κ by going to a weak field of gravity limit and demanding that $eq^n(23)$ reduces to Newton's EOM. This gives $\kappa = 8\pi$ (see [4]).

SYMMETRIES OF SPACETIME

In General Relativity (GR) we study or analyze the symmetries of a given spacetime from the concept of the so called Killing vectors. A Killing vector is any vector which satisfies the Killing's equation,

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \tag{24}$$

Let us understand how it is related to symmetries. Let γ be a geodesic curve on a spacetime with the curve parameter being $\lambda \in \mathbb{R}$; with $u^\alpha = \frac{dx^\alpha}{d\lambda}$ as the tangent vector and furthermore, let ξ be a Killing vector of the geometry. Then,

$$u^\beta \nabla_\beta (u^\alpha \xi_\alpha) = (u^\beta \nabla_\beta u^\alpha)\xi_\alpha + u^\beta (u^\alpha \nabla_\beta \xi_\alpha).$$

The first term in the above eq^n vanishes owing to the geodesic equation and the second term owing to the fact that $u^\beta u^\alpha$ is symmetric and $\nabla_\beta \xi_\alpha$ is antisymmetric in the indices. So, we get,

$$u^\beta \nabla_\beta (u^\alpha \xi_\alpha) = 0. \tag{25}$$

So it is clear from above that $u^\alpha \xi_\alpha$ is constant along γ. In case of timelike geodesics this will be a constant of motion. For example if you have a timelike Killing vector $u^\alpha \xi_\alpha$ will be energy (see [3]). So existence of a timelike Killing vector means that the spacetime is stationary (has time translational invariance recall *Noether's theorem*).

MAXIMALLY SYMMETRIC SPACES

A maximally symmetric space is a space which has the maximum symmetries possible for its dimension or technically one that has $n(n+1)/2$ linearly independent Killing vectors (in $n \in \mathbb{N}$ dimensions). We will see how this number comes about. First let us understand what a maximally symmetric space is (we will suppress the

time dimension for now).

Let us first prove a small lemma. Let ξ^μ be a Killing vector. Then (from $eq^n(14)$),

$$\begin{aligned}
(\nabla_\mu\nabla\nu - \nabla_\nu\nabla_\mu)\,\xi_\lambda &= -R^\rho_{\lambda\mu\nu}\xi_\rho,\\
\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu &= 0\ (from\ eq^n(24)),\\
leading\ to,\ \nabla_\lambda\nabla_\mu\xi_\nu + \nabla_\lambda\nabla_\nu\xi_\mu &= 0,\\
leading\ to,\ \nabla_\mu\nabla_\nu\xi_\lambda + \nabla_\nu\nabla_\lambda\xi_\mu + R^\rho_{\lambda\mu\nu}\xi_\rho &= 0\ ...a),\\
cyclic\ permutation,\ \nabla_\lambda\nabla_\mu\xi_\nu + \nabla_\mu\nabla_\nu\xi_\lambda + R^\rho_{\nu\lambda\mu}\xi_\rho &= 0\ ...b),\\
cyclic\ permutation,\ \nabla_\nu\nabla_\lambda\xi_\mu + \nabla_\lambda\nabla_\mu\xi_\nu + R^\rho_{\mu\nu\lambda}\xi_\rho &= 0\ ...c).
\end{aligned}$$

Now $a) - b) + c)$ gives,

$$\begin{aligned}
2\nabla_\nu\nabla_\lambda\xi_\mu + (R^\rho_{\mu\nu\lambda} + R^\rho_{\lambda\mu\nu} - R^\rho_{\nu\lambda\mu})\xi_\rho &= 0,\\
leading\ to,\ 2\nabla_\nu\nabla_\lambda\xi_\mu - 2R^\rho_{\nu\lambda\mu}\xi_\rho &= 0\ (by\ eq^n(18)).
\end{aligned}$$

Thus, we arrive at the following Killing vector lemma,

$$\nabla_\nu\nabla_\lambda\xi_\mu = R^\rho_{\nu\lambda\mu}\xi_\rho. \tag{26}$$

$Eq^n(26)$ is a 2^{nd} order differential equation and given $\xi_\mu(x_0)$ and $\nabla_\mu\xi_\nu(x_0)$ one can determine the Killing vector completely. Now, if your space has n dimension then the tangent space (the vector space at every point) at every point also has n dimension. Thus, at x_0 we have (linearly independent),

$$\#\xi_\mu(x_0) = n, \tag{27}$$

$$\#\nabla_\mu\xi_\nu(x_0) = n(n-1)/2\ (due\ to\ anti-symmetry). \tag{28}$$

Hence total no. of linearly independent Killing vectors which the space can admit is $n(n+1)/2$. Now if your space admits $n(n+1)/2$ Killing vectors then it is known to be maximally symmetric.

Homogeneous spaces

A homogeneous space is a space in which every point can be taken to every other point by an infinitesimal translation (a space where every point is equivalent) or technically a space which admits n linearly independent translational Killing vectors is called a homogeneous space (as existence of translation Killing vector means translational symmetry).

Isotropic spaces

An isotropic space is a space where $\forall$ *points* x_0, there exists an infinitesimal rotation at x_0 which fixes x_0 and takes any vector at x_0 to any other vector at x_0. This means if we have $n(n-1)/2$ Killing vectors ξ_μ such that $\xi_\mu(x_0) = 0$ and $\nabla_\mu\xi_\nu(x_0) \neq 0$ then the space is isotropic (which means the space is rotationally invariant around every point).

Then, a space which is maximally symmetric is homogeneous and isotropic and, vice-versa.

Isotropic implies homogeneity

Let ξ_μ and and $\tilde{\xi}_\mu$ be two rotational Killing vectors such that,

$$\begin{aligned}
&\xi_\mu(x) = 0,\\
&\tilde{\xi}_\mu(x+dx) = 0,\\
&\nabla_\mu\xi_\nu(x) \neq 0,\\
&\nabla_\mu\tilde{\xi}_\nu(x+dx) \neq 0.
\end{aligned}$$

Now consider,

$$M_\mu = \xi_\mu - \tilde{\xi}_\mu. \tag{29}$$

At x,

$$\begin{aligned}
M_\mu(x) &= \xi_\mu(x) - \tilde{\xi}_\mu(x+dx-dx)\\
&= -[\tilde{\xi}_\mu(x+dx) - dx^\lambda\nabla_\lambda\tilde{\xi}_\mu(x+dx)]\\
&= dx^\lambda\nabla_\lambda\tilde{\xi}_\mu(x+dx) \neq 0.
\end{aligned}$$

Also,

$$\nabla_\nu M_\mu(x) = dx^\lambda\nabla_\nu\nabla_\lambda\tilde{\xi}_\mu(x+dx) \propto \tilde{\xi}_\mu(x+dx) \neq 0.$$

Thus we get $M_\mu(x)$ to be a translational Killing vector made out of the difference between two rotational Killing vector and in this way one can proceed and argue that we can recover n linearly independent translational Killing vectors in an isotropic space. Thus, if a space is isotropic it is maximally symmetric.

Curvature of a maximally symmetric space

Suppose a space is maximally symmetric. Choose x_0 and choose a Riemann normal coordinate system there. Then, $\Gamma^\phi_{\alpha\beta}(x_0) = 0$ and $g_{\mu\nu}(x_0) = \eta_{\mu\nu}(x_0)$. Also, since there is rotational symmetry at every point in this space due to it being isotropic, at x_0 we have $R_{\alpha\beta\rho\sigma}$ to be Lorentz invariant. The only Lorentz invariant tensors in 4 dimension are $\eta_{\mu\nu}$ and $\epsilon_{\alpha\beta\rho\sigma}$ (which is the Levi-Civita tensor). Thus the Riemann tensor must be borne out of a linear combination of these. Hence,

$$R_{\alpha\beta\rho\sigma} = a\eta_{\alpha\beta}\eta_{\rho\sigma} + b\eta_{\alpha\rho}\eta_{\beta\sigma} + c\eta_{\alpha\sigma}\eta_{\beta\rho} + d\epsilon_{\alpha\beta\rho\sigma}. \tag{30}$$

Now use the symmetries of the Riemann tensor to determine the relationship between a, b, c, d which is,

$$a = d = b + c = 0. \tag{31}$$

Thus,

$$R_{\alpha\beta\rho\sigma} = b(\eta_{\alpha\rho}\eta_{\beta\sigma} - \eta_{\alpha\sigma}\eta_{\beta\rho}), \tag{32}$$

which in an arbitrary coordinate system is,

$$R_{\alpha\beta\rho\sigma}(x_0) = b(x_0)(g_{\alpha\rho}(x_0)g_{\beta\sigma}(x_0) - g_{\alpha\sigma}(x_0)g_{\beta\rho}(x_0)).$$

Since the space is isotropic we have,

$$R_{\alpha\beta\rho\sigma}(x) = b(x_0)(g_{\alpha\rho}(x)g_{\beta\sigma}(x) - g_{\alpha\sigma}(x)g_{\beta\rho}(x)). \quad (33)$$

Now the Ricci tensor becomes,

$$R_{\beta\sigma} = b(x)(n-1)g_{\beta\sigma}. \quad (34)$$

The Ricci scalar becomes,

$$R = n(n-1)b(x). \quad (35)$$

Now,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = g_{\mu\nu}b(x)(n-1)(1-n/2),$$
$$leading\ to,\ \nabla^\nu b(x) = \partial^\nu b(x) = 0\ (by\ eq^n(20)).$$

Thus b is a constant say κ which implies that in 3 spatial dimension,

$$R = 6\kappa. \quad (36)$$

So, we obtain the very important result that for maximally symmetric spaces the spatial curvature is constant.

Metric for maximally symmetric space

To find out the metric (or its form) note that we can assume that it is spherically symmetric [6] (as this symmetry is a subset). So, we have,

$$ds^2_M = B(r)dr^2 + r^2 d\Omega^2_{n-1}, \quad (37)$$

where $B(r)$ is any arbitrary smooth function of r and $d\Omega_{n-1}$ is the usual metric on S^{n-1} where n is the dimension of the space. $Eq^n(37)$ is the most general form of a spherically symmetric metric[3] (see [5]).

Now let $n = 3$ and with this compute $R_{\mu\nu}$ from ds^2_M and equate it with $R_{\mu\nu} = \kappa(n-1)g_{\mu\nu}$[4]. You should obtain,

$$R_{rr} = \frac{B'}{rB} = 2\kappa B,$$
$$R_{\theta\theta} = 1 - \frac{1}{B} + \frac{rB'}{2B^2} = 2kr^2,$$
$$R_{\phi\phi} = sin^2\theta\ R_{\theta\theta}.$$

This leads to,

$$B(r) = \frac{1}{1-\kappa r^2}. \quad (38)$$

Thus the metric becomes,

$$ds^2_M = \frac{1}{1-\kappa r^2}dr^2 + r^2 d\Omega^2_2. \quad (39)$$

Scaling the metric

Replace $\kappa \leftrightarrow \kappa/L^2$ in $eq^n(39)$ to get a new metric as,

$$\tilde{ds}^2_M = \frac{dr^2}{1-\frac{\kappa}{L^2}r^2} + r^2 d\Omega^2_2.$$

Now define $\tilde{r} = r/L$, then,

$$\tilde{ds}^2_M = L^2 ds^2_M. \quad (40)$$

Thus, $\kappa \leftrightarrow \kappa/L^2 \ \Leftrightarrow\ ds^2_M \leftrightarrow L^2 ds^2_M$. Hence, scaling κ results in overall scaling of the metric. So, $\kappa = 0, \pm 1$ (as its scaling doesn't matter).

$\kappa = 0$ implies flat

Clearly, if $\kappa = 0$ we obtain the flat metric for our universe.

$\kappa = +1$ implies metric on a sphere

Consider a 3-sphere S^3 embedded in $\mathbb{R}^4$ (with coordinates (x, y, z, u)). Then (equation of a sphere),

$$x^2 + y^2 + z^2 + u^2 = 1,$$
$$leading\ to,\ u^2 = 1 - (x^2 + y^2 + z^2) = 1 - r^2,$$
$$leading\ to,\ du = -\frac{rdr}{\sqrt{1-r^2}}.$$

Now the metric on S^3 embedded in $\mathbb{R}^4$ is given as,

$$ds^2 = dx^2 + dy^2 + dz^2 + du^2$$
$$(by\ Pythagoras'\ theorem),$$
$$leading\ to,\ ds^2 = \frac{dr^2}{1-r^2} + d\Omega^2_2.$$

So $\kappa = +1$ means we have the metric on S^3.

$\kappa = -1$ implies metric on a hyperboloid

Consider a hyperboloid H^3 embedded in $\mathbb{R}^{1,4}$ (with coordinates (x, y, z, u))[5]. Then (equation of a hyper-

[3] For now just observe that in 3 dimension $d\Omega^2_2 = d\theta^2 + sin^2\theta\ d\phi^2$ is the metric on S^2 and hence is spherically symmetric. S^2 is the unit sphere.

[4] You can compute $R_{\mu\nu}$ using this formula,

$$R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\rho\mu,\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}$$

[5] H^3 can't be embedded in $\mathbb{R}^4$ because it contains mixed signs in its metric and hence the metric is not positive-definite.

boloid),

$$u^2 - x^2 - y^2 - z^2 = 1,$$
$$\textit{leading to, } u^2 = 1 + (x^2 + y^2 + z^2) = 1 + r^2,$$
$$\textit{leading to, } du = \frac{rdr}{\sqrt{1+r^2}}.$$

Now the metric on S^3 embedded in $\mathbb{R}^4$ is given as,

$$ds^2 = -du^2 + dy^2 + dz^2 + du^2$$
$$(\textit{Minkowski metric on } \mathbb{R}^4),$$
$$\textit{leading to, } ds^2 = \frac{dr^2}{1+r^2} + d\Omega_2^2.$$

So $\kappa = -1$ means we have the metric on H^3.

FLRW METRIC

We assume that at the level of galaxies our universe is spatially homogeneous and isotropic and this is known as the *Cosmological Principle*. So, it being maximally symmetric has its spatial part proportional to the metric as expressed in $eq^n(39)$. So, the spatial part of FLRW metric becomes,

$$ds^2_{spatial} = a^2(t)\left(\frac{1}{1-\kappa r^2}dr^2 + r^2 d\Omega_2^2\right).$$

where $a(t)$ is the time-dependent proportionality factor (known as the scale function of the universe, with which the universe scales in time). The complete metric then becomes (the temporal part being just dt^2 which means at different locations the clocks would tick at the same rate which is true from spatial homogeneity and isotropy, also one can easily see with $\kappa = 0$, dt^2 as the temporal term gives the flat metric),

$$ds^2_{FLRW} = -dt^2 + a^2(t)\left(\frac{1}{1-\kappa r^2}dr^2 + r^2 d\Omega_2^2\right). \quad (41)$$

with $\kappa = 0, \pm 1$ where $\kappa = 0$ implies our universe is flat, $\kappa = +1$ implies our universe is like a sphere and $\kappa = -1$ implies our universe is like a saddle (hyperboloid). The metric above is written from the perspectives of a comoving observer in comoving coordinates (t, r, θ, ϕ).

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