Three-Character MLDEs in the j-Plane: Wronskian Structure and Index Constraints

Anxu GAO

Supervisor: Arpit Das

Abstract

In the Mathur–Mukhi–Sen (MMS) classification scheme for rational conformal field theories (RCFTs), theories and their associated modular linear differential equations (MLDEs) are characterized by a pair of non-negative integers $[n, \ell]$, where n is the number of characters and ℓ is the Wronskian index. An alternative way to study MLDEs is to reformulate them in terms of the modular invariant j, where their singularity structure can be analysed directly [1]. In this work, we develop this j-space approach for third-order MLDEs associated with three-character systems. We derive an all-order formula converting iterated modular-covariant derivatives in the τ-plane into ordinary differential operators in j, and use it together with a local Wronskian analysis to construct the corresponding j-space MLDEs. For irreducible three-character systems, we show that $\ell \in 3\mathbb{Z}$, so that the allowed residue classes are $\ell = 6r$ and $\ell = 6r+3$. We further derive the generic j-space equations for these classes and their associated local no-logarithm constraints.

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1 Introduction

Modular linear differential equations (MLDEs) provide a useful framework for studying the characters of rational conformal field theories (RCFTs). The characters form a finite-dimensional

vector-valued modular function and transform among themselves under the modular group. As a consequence of this modular structure, a system of n linearly independent characters can be described by an n-th order differential equation whose coefficients are modular functions. The order of the MLDE therefore corresponds to the number of independent characters, while its modular properties impose strong constraints on the possible solutions.

An important quantity in this approach is the Wronskian index ℓ, which measures the zeros of the Wronskian of the character solutions in the modular fundamental domain. Together with the number of characters, the value of ℓ constrains the form of the MLDE and the possible exponents of its solutions. This has made the classification of MLDEs with fixed (n, ℓ) a useful approach to the classification of candidate RCFT character vectors; see, for example, [2–4].

Most MLDEs are naturally formulated in the modular parameter τ, where modular covariance is manifest. For the analysis of their singularities, however, it is useful to pass to the modular invariant $j = j(\tau)$. The elliptic points of the modular fundamental domain are then mapped to $j = 0$ and $j = 1728$, while zeros of the Wronskian away from the elliptic points appear as movable singularities in the resulting ordinary differential equation. This formulation allows the local exponent structure, the pole orders of the coefficient functions, and the apparent-singularity conditions to be studied directly using ordinary differential equations.

2 Three-character MLDEs in the j-plane

We first derive an all-order formula for converting iterated modular-covariant derivatives in the τ-plane into ordinary derivatives in the j-plane.

Then, we will apply the resulting formula and a local Wronskian analysis to third-order MLDEs. On the generic locus of distinct simple movable Wronskian zeros, we obtain the j-plane singularity structure for the two irreducible residue classes

$$\ell = 6r, \qquad \ell = 6r + 3, \qquad r \in \mathbb{Z}_{\geq 0}, \tag{2.1}$$

and prove that these are the only possible classes for an irreducible three-character system [1].

The irreducibility qualification in the last statement is essential. A reducible three-component character vector can have $\ell = 2$ or $\ell = 4$; the familiar examples are obtained by adjoining a constant character to a two-character system (see [3, 4] for more details). Thus the result is not that every third-order MLDE has $\ell \in 3\mathbb{Z}$, but rather that every genuine rank-three system whose modular representation is irreducible has $\ell \in 3\mathbb{Z}$. Throughout the divisibility argument, the characters are assumed linearly independent and non-logarithmic. They carry a genuine weight-zero representation of $SL(2, \mathbb{Z})$; in Subsection 2.9, linear independence will imply that $-I_2$ acts trivially, so that this representation descends to $PSL(2, \mathbb{Z})$. Non-logarithmicity ensures that the cusp exponents determine the eigenvalues of T.

[1] A different approach to proving the above result which uses representation of $PSL(2, \mathbb{Z})$ was done in [5]. Here, on the contrary, we shall follow a more direct, explicit and a different approach.

2.1 Conventions and elementary modular identities

We use

$$\vartheta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}, \qquad \mathcal{D}_k := \vartheta - \frac{k}{12}E_2, \tag{2.2}$$

where $\mathcal{D}_k$ maps a modular object of weight k to one of weight $k+2$. The iterated derivative acting on a weight-zero function is

$$\mathcal{D}^{(n)} := \mathcal{D}_{2n-2} \circ \mathcal{D}_{2n-4} \circ \cdots \circ \mathcal{D}_2 \circ \mathcal{D}_0, \qquad \mathcal{D}^{(0)} := 1. \tag{2.3}$$

We take the standard normalization

$$j = \frac{E_4^3}{\Delta}, \qquad j - 1728 = \frac{E_6^2}{\Delta}, \qquad \Delta = \frac{E_4^3 - E_6^2}{1728}. \tag{2.4}$$

The Ramanujan identities imply

$$\mathcal{D}_4 E_4 = -\frac{1}{3}E_6, \qquad \mathcal{D}_6 E_6 = -\frac{1}{2}E_4^2. \tag{2.5}$$

It is convenient to introduce

$$J(\tau) := \vartheta j = -j\frac{E_6}{E_4}. \tag{2.6}$$

Using (2.4), we immediately find

$$J^2 = j^2\frac{E_6^2}{E_4^2} = E_4 j(j-1728), \qquad J^3 = -E_6 j^2(j-1728). \tag{2.7}$$

We will also need the weight-two covariant derivative of J. Since E_6/E_4 has weight two, the covariant quotient rule gives

$$\begin{aligned}\mathcal{D}_2\left(\frac{E_6}{E_4}\right) &= \frac{(\mathcal{D}_6 E_6)E_4 - E_6(\mathcal{D}_4 E_4)}{E_4^2} \\ &= -\frac{1}{2}E_4 + \frac{1}{3}\frac{E_6^2}{E_4^2}.\end{aligned} \tag{2.8}$$

Therefore

$$\begin{aligned}\mathcal{D}_2 J &= -\mathcal{D}_2\left(j\frac{E_6}{E_4}\right) \\ &= -J\frac{E_6}{E_4} - j\mathcal{D}_2\left(\frac{E_6}{E_4}\right) \\ &= j\left(\frac{2}{3}\frac{E_6^2}{E_4^2} + \frac{1}{2}E_4\right) \\ &= E_4\left(\frac{2(j-1728)}{3} + \frac{j}{2}\right).\end{aligned} \tag{2.9}$$

Dividing by J^2 motivates the rational function

$$M(j) := \frac{\mathcal{D}_2 J}{J^2} = \frac{2}{3j} + \frac{1}{2(j-1728)}. \tag{2.10}$$

2.2 The all-order τ–j conversion

Let f be a weight-zero function that depends on τ only through $j(\tau)$. The first derivative is simply

$$\mathcal{D}^{(1)} f = \vartheta f = J\,\partial_j f. \tag{2.11}$$

At second order,

$$\begin{aligned}\mathcal{D}^{(2)} f &= \mathcal{D}_2(J\partial_j f)\\ &= (\mathcal{D}_2 J)\partial_j f + J\vartheta(\partial_j f)\\ &= (\mathcal{D}_2 J)\partial_j f + J^2\partial_j^2 f\\ &= J^2\left(\partial_j^2 + M\partial_j\right) f.\end{aligned} \tag{2.12}$$

Equivalently,

$$\begin{aligned}\mathcal{D}^{(2)} &= E_4\left[j(j-1728)\partial_j^2 + \left(\frac{j}{2} + \frac{2(j-1728)}{3}\right)\partial_j\right]\\ &= J^2\left[\partial_j^2 + \left(\frac{1}{2(j-1728)} + \frac{2}{3j}\right)\partial_j\right].\end{aligned} \tag{2.13}$$

At third order,

$$\begin{aligned}\mathcal{D}^{(3)} f &= \mathcal{D}_4\Big[J^2\left(\partial_j^2 + M\partial_j\right) f\Big]\\ &= J^3\left(\partial_j + 2M\right)\left(\partial_j^2 + M\partial_j\right) f\\ &= J^3\Big[\partial_j^3 + 3M\partial_j^2 + (M' + 2M^2)\partial_j\Big] f,\end{aligned} \tag{2.14}$$

where a prime denotes d/dj. From (2.10),

$$3M = \frac{2}{j} + \frac{3}{2(j-1728)}, \qquad M' + 2M^2 = \frac{2}{9j^2} + \frac{4}{3j(j-1728)}. \tag{2.15}$$

Consequently

$$\boxed{\mathcal{D}^{(3)} = J^3\left[\partial_j^3 + \left(\frac{2}{j} + \frac{3}{2(j-1728)}\right)\partial_j^2 + \left(\frac{2}{9j^2} + \frac{4}{3j(j-1728)}\right)\partial_j\right].} \tag{2.16}$$

The pattern has a simple all-order form. Define differential operators $\mathcal{P}_n$ by

$$\mathcal{D}^{(n)} f(j) = J^n \mathcal{P}_n f(j), \qquad \mathcal{P}_0 = 1, \qquad \mathcal{P}_1 = \partial_j. \tag{2.17}$$

Suppose (2.17) holds at order n. Then

$$\begin{aligned}\mathcal{D}^{(n+1)} f &= \mathcal{D}_{2n}(J^n \mathcal{P}_n f)\\ &= \vartheta(J^n \mathcal{P}_n f) - \frac{n}{6} E_2 J^n \mathcal{P}_n f\\ &= n J^{n-1}\left(\vartheta J - \frac{1}{6} E_2 J\right)\mathcal{P}_n f + J^{n+1}\partial_j(\mathcal{P}_n f)\\ &= J^{n+1}\left(\partial_j + nM\right)\mathcal{P}_n f.\end{aligned} \tag{2.18}$$

Thus

$$\boxed{\mathcal{P}_{n+1} = (\partial_j + nM)\circ \mathcal{P}_n,} \tag{2.19}$$

and hence

$$\boxed{\mathcal{D}^{(n)} = J^n\big(\partial_j + (n-1)M\big)\circ\big(\partial_j + (n-2)M\big)\circ\cdots\circ\big(\partial_j + M\big)\circ\partial_j.} \tag{2.20}$$

The ordering displayed in (2.20) is part of the formula: every ∂_j acts on all coefficient functions to its right. Therefore the factors cannot be treated as commuting objects, which is why a commuting binomial reading is not valid.

For reference, the first nontrivial terms of the general expansion are

$$\begin{aligned}\mathcal{P}_n = \partial_j^n &+ \binom{n}{2} M\,\partial_j^{n-1} \\ &+ \left[\binom{n}{3} M' + \frac{n(n-1)(n-2)(3n-1)}{24} M^2\right]\partial_j^{n-2} + \cdots .\end{aligned} \tag{2.21}$$

For example,

$$\begin{aligned}\mathcal{P}_4 = \partial_j^4 &+ 6M\partial_j^3 + (4M' + 11M^2)\partial_j^2 \\ &+ (M'' + 7MM' + 6M^3)\partial_j .\end{aligned} \tag{2.22}$$

2.3 Third-order Wronskians and coefficient functions

Consider a monic third-order equation in the j-plane,

$$\big(\partial_j^3 + \psi_2(j)\partial_j^2 + \psi_4(j)\partial_j + \psi_6(j)\big)\,\chi(j) = 0, \tag{2.23}$$

with three linearly independent solutions χ_0, χ_1, χ_2. Define the four minors

$$W_3 := \det\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 \\ \chi_0' & \chi_1' & \chi_2' \\ \chi_0'' & \chi_1'' & \chi_2'' \end{pmatrix}, \qquad W_2 := \det\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 \\ \chi_0' & \chi_1' & \chi_2' \\ \chi_0''' & \chi_1''' & \chi_2''' \end{pmatrix},$$

$$W_1 := \det\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 \\ \chi_0'' & \chi_1'' & \chi_2'' \\ \chi_0''' & \chi_1''' & \chi_2''' \end{pmatrix}, \qquad W_0 := \det\begin{pmatrix} \chi_0' & \chi_1' & \chi_2' \\ \chi_0'' & \chi_1'' & \chi_2'' \\ \chi_0''' & \chi_1''' & \chi_2''' \end{pmatrix}. \tag{2.24}$$

Cramer's rule gives

$$\psi_2 = -\frac{W_2}{W_3} = -\partial_j \log W_3, \qquad \psi_4 = \frac{W_1}{W_3}, \qquad \psi_6 = -\frac{W_0}{W_3}. \tag{2.25}$$

The determinant containing the rows (χ, χ', χ''') is therefore W_2, not W_1.

Let x be a local coordinate and suppose, at leading order,

$$\chi_a(x) \sim x^{\lambda_a}, \qquad a = 0, 1, 2. \tag{2.26}$$

For the determinant formulas below, assume first that $\lambda_0, \lambda_1, \lambda_2$ are pairwise distinct. Introduce the elementary symmetric polynomials

$$s_1 := \lambda_0 + \lambda_1 + \lambda_2, \qquad s_2 := \lambda_0\lambda_1 + \lambda_0\lambda_2 + \lambda_1\lambda_2, \qquad s_3 := \lambda_0\lambda_1\lambda_2. \tag{2.27}$$

The numerical determinant controlling the leading Wronskian coefficient is

$$V_\lambda := \det\begin{pmatrix} 1 & 1 & 1 \\ \lambda_0 & \lambda_1 & \lambda_2 \\ \lambda_0(\lambda_0-1) & \lambda_1(\lambda_1-1) & \lambda_2(\lambda_2-1) \end{pmatrix} \\ = (\lambda_1-\lambda_0)(\lambda_2-\lambda_0)(\lambda_2-\lambda_1). \tag{2.28}$$

This is called the Vandermonde factor. It is nonzero precisely when the three exponents are pairwise distinct. If two leading exponents coincide, the leading determinant vanishes, so one cannot insert the repeated exponents directly into the formulas below. One must first take suitable constant linear combinations of the local solutions and determine their actual first nonzero powers.

Directly factoring the powers of x from the rows and columns gives

$$W_3 \sim V_\lambda\, x^{s_1-3}, \tag{2.29}$$
$$W_2 \sim V_\lambda(s_1-3)x^{s_1-4}, \tag{2.30}$$
$$W_1 \sim V_\lambda(s_2+1-s_1)x^{s_1-5}, \tag{2.31}$$
$$W_0 \sim V_\lambda s_3 x^{s_1-6}. \tag{2.32}$$

For example, (2.32) follows by factoring one λ_a from each column of the determinant with derivative orders $(1,2,3)$. Equations (2.29)– (2.32) also reproduce the local indicial polynomial

$$\lambda(\lambda-1)(\lambda-2)+p_{-1}\lambda(\lambda-1)+q_{-2}\lambda+r_{-3}=0, \tag{2.33}$$

where

$$p_{-1}=3-s_1, \qquad q_{-2}=s_2+1-s_1, \qquad r_{-3}=-s_3. \tag{2.34}$$

In particular, the assertion that the leading coefficient in ψ_4 is always just s_2 is not correct: the general answer is s_2+1-s_1. It reduces to s_2 only when $s_1=1$.

When one of the λ_a vanishes, the leading term in (2.32) vanishes. One must then keep sufficiently many terms in the Frobenius series before assigning an order to W_0. This point will be important at the elliptic points.

2.4 Relating the j- and τ-Wronskians

Let W_j denote W_3 in (2.24), and let W_τ denote the Wronskian whose three rows have entries χ_a, $\mathcal{D}^{(1)}\chi_a$, and $\mathcal{D}^{(2)}\chi_a$, respectively. From (2.11) and (2.12), a row operation removes the $M\partial_j$ term from the last row, after which one can factor J and J^2 from the second and third rows. Hence

$$\boxed{W_\tau = J^3 W_j.} \tag{2.35}$$

Let the local j-exponents at $j=0$, $j=1728$, and a generic point $j=p$ be denoted by $\lambda_a^{(\rho)}$, $\lambda_a^{(i)}$, and $\lambda_a^{(p)}$, respectively. Near ρ,

$$j \sim (\tau-\rho)^3, \qquad J \sim j^{2/3}. \tag{2.36}$$

Combining (2.29) and (2.35),

$$W_\tau \sim j^{\lambda_0^{(\rho)}+\lambda_1^{(\rho)}+\lambda_2^{(\rho)}-1}. \tag{2.37}$$

If ℓ_ρ is the contribution of this point to the Wronskian index, then $W_\tau \sim j^{\ell_\rho/6}$, and therefore

$$\boxed{\lambda_0^{(\rho)}+\lambda_1^{(\rho)}+\lambda_2^{(\rho)}-1=\frac{\ell_\rho}{6}.} \tag{2.38}$$

Near i, with $x = j-1728$,

$$x \sim (\tau-i)^2, \qquad J \sim x^{1/2}. \tag{2.39}$$

It follows that

$$W_\tau \sim x^{\lambda_0^{(i)}+\lambda_1^{(i)}+\lambda_2^{(i)}-3/2}, \tag{2.40}$$

and hence

$$\boxed{\lambda_0^{(i)}+\lambda_1^{(i)}+\lambda_2^{(i)}-\frac{3}{2}=\frac{\ell_i}{6}.} \tag{2.41}$$

At a generic point $j = p$, J is finite and nonzero. Thus

$$\boxed{\lambda_0^{(p)}+\lambda_1^{(p)}+\lambda_2^{(p)}-3=\frac{\ell_p}{6}.} \tag{2.42}$$

For $\ell = 6r$ with no Wronskian zero at either elliptic point, regularity and the absence of logarithms give the minimal local exponent sets

$$\left\{0,\frac{1}{3},\frac{2}{3}\right\}_{j=0}, \qquad \left\{0,\frac{1}{2},1\right\}_{j=1728}. \tag{2.43}$$

At an isolated simple movable zero, $\ell_p = 6$, so (2.42) gives a sum equal to four. The unique strictly increasing triple of nonnegative integers with this sum is

$$\boxed{\{\lambda_0^{(p)},\lambda_1^{(p)},\lambda_2^{(p)}\}=\{0,1,3\}.} \tag{2.44}$$

To justify the strict inequalities, let $x = j-p$. Since p is an ordinary point, the three characters are analytic and single-valued there and have convergent Taylor expansions

$$\chi_a(x)=\sum_{n=0}^{\infty} c_{a,n}x^n.$$

By taking suitable constant linear combinations of these solutions, or equivalently by performing Gaussian elimination on their Taylor coefficients, one can choose a local basis of the form

$$\widetilde{\chi}_a(x)=x^{n_a}\big(c_a+O(x)\big), \qquad c_a\neq 0, \qquad 0\le n_0<n_1<n_2.$$

Thus the local exponents are strictly increasing nonnegative integers. Their sum is fixed by (2.42) to be four. The only possibility is

$$(n_0,n_1,n_2)=(0,1,3).$$

This is the third-order analogue of the pair $(0,2)$ at a simple apparent singularity of a second-order MLDE (see [1] for more details).

For later use, the generic local orders for (2.43) and (2.44) are

point	local exponents	W_3	W_1	W_0
$j=0$	$0,\frac{1}{3},\frac{2}{3}$	j^{-2}	j^{-4}	j^{-4}
$j=1728$	$0,\frac{1}{2},1$	$x^{-3/2}$	$x^{-5/2}$	$x^{-5/2}$
$j=p$	$0,1,3$	x	1	1

where $x=j-1728$ or $x=j-p$ in the corresponding row. Let us explain the entry $W_0\sim x^{-5/2}$ at $j=1728$, since it corrects an important power counting error. The naive power in (2.32) is $s_1-6=-9/2$, but its coefficient contains $s_3=0$. Any contribution with total Frobenius shift one either duplicates the existing exponent 1 or leaves a vanishing determinant. The first generic nonzero determinant therefore has total shift two (for example through the x^2 correction to the exponent-zero solution), which gives $-9/2+2=-5/2$. It is not $-7/2$.

The table and (2.25) imply

$$\psi_4\sim\begin{cases} j^{-2}, & j\to 0,\\ (j-1728)^{-1}, & j\to 1728,\\ (j-p)^{-1}, & j\to p,\end{cases} \tag{2.45}$$

$$\psi_6\sim\begin{cases} j^{-2}, & j\to 0,\\ (j-1728)^{-1}, & j\to 1728,\\ (j-p)^{-1}, & j\to p.\end{cases} \tag{2.46}$$

In particular, ψ_6 has only a simple pole at $j=1728$ in the $\ell=6r$ case.

2.5 The generic simple-zero $\ell=6r$ equation

Let the r distinct simple movable zeros of the τ-Wronskian be $p_1,\dots,p_r$, and define

$$P_r(j):=\prod_{I=1}^{r}(j-p_I). \tag{2.47}$$

In this subsection $p_I\notin\{0,1728\}$, no two p_I coincide, and there are no additional Wronskian zeros at the elliptic points. Coincident zeros or extra elliptic zeros require the corresponding multiplicities in the local formulas and are not described by the factored simple-root parametrization below. For $\ell=6r$, the j-Wronskian has the form

$$W_j\propto j^{-2}(j-1728)^{-3/2}P_r(j). \tag{2.48}$$

Abel's identity therefore gives

$$\boxed{\psi_2(j)=\frac{2}{j}+\frac{3}{2(j-1728)}-\frac{P_r'(j)}{P_r(j)}=\frac{2}{j}+\frac{3}{2(j-1728)}-\sum_{I=1}^{r}\frac{1}{j-p_I}.} \tag{2.49}$$

Let the three q-exponents at the cusp be $\alpha_0,\alpha_1,\alpha_2$, and set

$$\sigma_1:=\sum_{a=0}^{2}\alpha_a,\qquad \sigma_2:=\sum_{0\le a<b\le 2}\alpha_a\alpha_b,\qquad \sigma_3:=\alpha_0\alpha_1\alpha_2. \tag{2.50}$$

The valence formula gives

$$\sigma_1 = \frac{1}{2} - r. \tag{2.51}$$

Since $q = j^{-1} + O(j^{-2})$, the exponents at $j = \infty$ are $-\alpha_a$. Write

$$\psi_2 \sim \frac{A_\infty}{j}, \qquad \psi_4 \sim \frac{B_\infty}{j^2}, \qquad \psi_6 \sim \frac{C_\infty}{j^3}. \tag{2.52}$$

Substitution of $\chi \sim j^\beta$ into (2.23) gives

$$\beta(\beta-1)(\beta-2) + A_\infty \beta(\beta-1) + B_\infty \beta + C_\infty = 0. \tag{2.53}$$

The roots are $\beta_a = -\alpha_a$. Comparing elementary symmetric polynomials yields

$$A_\infty = 3 + \sigma_1, \qquad B_\infty = \sigma_2 + 1 + \sigma_1, \qquad C_\infty = \sigma_3. \tag{2.54}$$

For (2.51), this becomes

$$A_\infty = \frac{7}{2} - r, \qquad B_\infty = \sigma_2 + \frac{3}{2} - r, \qquad C_\infty = \sigma_3. \tag{2.55}$$

The local pole orders in (2.45)– (2.46), together with (2.55), determine the rational form on this locus [2],

$$\psi_4(j) = \frac{Q_{4,r+1}(j)}{j^2(j-1728)P_r(j)}, \qquad \deg Q_{4,r+1} \le r+1, \qquad [j^{r+1}]Q_{4,r+1} = \sigma_2 + \frac{3}{2} - r, \tag{2.56}$$

$$\psi_6(j) = \frac{Q_{6,r}(j)}{j^2(j-1728)P_r(j)}, \qquad \deg Q_{6,r} \le r, \qquad [j^r]Q_{6,r} = \sigma_3. \tag{2.57}$$

On the generic locus where the displayed leading coefficients are nonzero, we may factor the two polynomials and write

$$\psi_4(j) = \left(\sigma_2 + \frac{3}{2} - r\right) \frac{\prod_{A=1}^{r+1}(j - b_{4,A})}{j^2(j-1728)\prod_{I=1}^{r}(j-p_I)}, \tag{2.58}$$

$$\psi_6(j) = \sigma_3 \frac{\prod_{A=1}^{r}(j - b_{6,A})}{j^2(j-1728)\prod_{I=1}^{r}(j-p_I)}. \tag{2.59}$$

[2] Below, we will use the notation: $[j^n]Q(j)$ to denote the coefficient multiplying j^n in $Q(j)$

Thus the corrected $\ell = 6r$ equation is

$$0 = \partial_j^3 \chi + \left[\frac{2}{j} + \frac{3}{2(j-1728)} - \sum_{I=1}^{r} \frac{1}{j-p_I} \right] \partial_j^2 \chi + \left(\sigma_2 + \frac{3}{2} - r \right) \frac{\prod_{A=1}^{r+1} (j - b_{4,A})}{j^2 (j-1728) \prod_{I=1}^{r} (j-p_I)} \partial_j \chi + \sigma_3 \frac{\prod_{A=1}^{r} (j - b_{6,A})}{j^2 (j-1728) \prod_{I=1}^{r} (j-p_I)} \chi . \tag{2.60}$$

The root parametrization in (2.60) must be replaced by the polynomial form (2.56)–(2.57) when a leading coefficient vanishes.

The exponent set at $j = 0$ imposes the useful residue constraint

$$\lim_{j \to 0} j^2 \psi_4(j) = \frac{2}{9}. \tag{2.61}$$

For the factored form, this is

$$\left(\sigma_2 + \frac{3}{2} - r \right) \frac{\prod_{A=1}^{r+1} b_{4,A}}{1728 \prod_{I=1}^{r} p_I} = \frac{2}{9}. \tag{2.62}$$

The movable points are apparent singularities and obey further no-log conditions derived below.

2.6 Direct specialization to the $(3,6)$ MLDE

Consider the τ-plane equation

$$\left[\mathcal{D}^{(3)} + \frac{\mu_{2,1} E_4^2 E_6}{E_4^3 + \mu_{12} \Delta} \mathcal{D}^{(2)} + \frac{\mu_{4,0} E_4^4 + \mu_{1,12} E_4 \Delta}{E_4^3 + \mu_{12} \Delta} \mathcal{D}^{(1)} + \frac{\mu_{0,3} E_6^3 + \mu_{12,1} E_6 \Delta}{E_4^3 + \mu_{12} \Delta} \right] \chi = 0. \tag{2.63}$$

The cusp indicial equation fixes

$$\mu_{2,1} = 1, \qquad \mu_{4,0} = \sigma_2 + \frac{1}{9}, \qquad \mu_{0,3} = -\sigma_3. \tag{2.64}$$

Introduce the τ-plane root parameters

$$a := -\mu_{12}, \qquad b_4^{(\tau)} := -\frac{\mu_{1,12}}{\mu_{4,0}}, \qquad b_6^{(\tau)} := 1728 - \frac{\mu_{12,1}}{\mu_{0,3}}, \tag{2.65}$$

whenever the corresponding rigid coefficient is nonzero. Since

$$E_4^3 + \mu_{12} \Delta = \Delta (j - a), \tag{2.66}$$

the three lower-order coefficients become

$$\frac{E_4^2 E_6}{E_4^3+\mu_{12}\Delta} = -\frac{J}{j-a}, \tag{2.67}$$

$$\frac{\mu_{4,0}E_4^4+\mu_{1,12}E_4\Delta}{E_4^3+\mu_{12}\Delta} = \mu_{4,0}E_4\frac{j-b_4^{(\tau)}}{j-a}, \tag{2.68}$$

$$\frac{\mu_{0,3}E_6^3+\mu_{12,1}E_6\Delta}{E_4^3+\mu_{12}\Delta} = \mu_{0,3}E_6\frac{j-b_6^{(\tau)}}{j-a}. \tag{2.69}$$

Using (2.12), (2.14), and (2.7), division by J^3 gives the exact equation

$$0 = \left[\mathcal{P}_3 - \frac{1}{j-a}\mathcal{P}_2 + \frac{\mu_{4,0}(j-b_4^{(\tau)})}{j(j-1728)(j-a)}\partial_j - \frac{\mu_{0,3}(j-b_6^{(\tau)})}{j^2(j-1728)(j-a)}\right]\chi. \tag{2.70}$$

Equivalently,

$$\begin{aligned} 0 = \chi''' &+ \left[\frac{2}{j} + \frac{3}{2(j-1728)} - \frac{1}{j-a}\right]\chi'' \\ &+ \left[\frac{2}{9j^2} + \frac{4}{3j(j-1728)} - \frac{M(j)}{j-a} + \frac{\mu_{4,0}(j-b_4^{(\tau)})}{j(j-1728)(j-a)}\right]\chi' \\ &- \frac{\mu_{0,3}(j-b_6^{(\tau)})}{j^2(j-1728)(j-a)}\chi. \end{aligned} \tag{2.71}$$

At $j=a$, the coefficient of χ'' has residue -1, whereas the other two coefficients have only simple poles. The indicial polynomial is therefore

$$\lambda(\lambda-1)(\lambda-2)-\lambda(\lambda-1) = \lambda(\lambda-1)(\lambda-3), \tag{2.72}$$

which confirms the exponents $(0,1,3)$.

The large-j limit of the coefficient of χ' in (2.71) is

$$\frac{1}{j^2}\left[\frac{14}{9} - \frac{7}{6} + \left(\sigma_2 + \frac{1}{9}\right)\right] = \frac{\sigma_2 + 1/2}{j^2}, \tag{2.73}$$

in agreement with (2.55) at $r=1$. Moreover, the last line of (2.71) has only one power of $(j-1728)$ in the denominator. This explicitly exhibits both corrections to the proposed handwritten equation: the ψ_4 normalization is $\sigma_2+1/2$, not σ_2, and ψ_6 has a simple rather than a double pole at $j=1728$.

2.7 The generic simple-zero realization of the second allowed class: $\ell = 6r+3$

The proof in subsection 2.9 below shows that an irreducible third-order system can also have $\ell = 6r+3$. In the generic distribution of Wronskian zeros, with distinct simple $p_I \notin \{0, 1728\}$ and no additional elliptic zeros, this class has

$$\ell_\rho = 0, \qquad \ell_i = 3, \qquad \ell_\tau = 6r. \tag{2.74}$$

The j-Wronskian is now

$$W_j \propto j^{-2}(j-1728)^{-1}P_r(j), \tag{2.75}$$

so Abel's identity gives

$$\boxed{\psi_2(j) = \frac{2}{j} + \frac{1}{j-1728} - \sum_{I=1}^{r} \frac{1}{j-p_I}.} \tag{2.76}$$

At $j=0$ the local exponents remain $(0,1/3,2/3)$. At $j=1728$, (2.41) gives a sum equal to two, and the canonical logarithm-free exponent set is

$$\left\{0, \frac{1}{2}, \frac{3}{2}\right\}_{j=1728}. \tag{2.77}$$

For this set, (2.34) gives

$$p_{-1} = 1, \qquad q_{-2} = -\frac{1}{4}, \qquad r_{-3} = 0. \tag{2.78}$$

Thus ψ_4 has a double pole of coefficient $-1/4$ at $j=1728$. The leading term of W_0 again vanishes because one exponent is zero, but now the first correction x^1 does not duplicate either $1/2$ or $3/2$. It follows that $W_0 \sim x^{-3}$, while $W_3 \sim x^{-1}$, and hence $\psi_6 \sim x^{-2}$.

The cusp valence formula gives

$$\sigma_1 = \frac{1}{2} - \frac{6r+3}{6} = -r. \tag{2.79}$$

Using (2.54),

$$A_\infty = 3-r, \qquad B_\infty = \sigma_2 + 1 - r, \qquad C_\infty = \sigma_3. \tag{2.80}$$

Consequently the rational form on this locus is

$$\psi_4(j) = \frac{\widehat{Q}_{4,r+2}(j)}{j^2(j-1728)^2 P_r(j)}, \qquad \deg \widehat{Q}_{4,r+2} \le r+2, \qquad [j^{r+2}]\widehat{Q}_{4,r+2} = \sigma_2 + 1 - r, \tag{2.81}$$

$$\psi_6(j) = \frac{\widehat{Q}_{6,r+1}(j)}{j^2(j-1728)^2 P_r(j)}, \qquad \deg \widehat{Q}_{6,r+1} \le r+1, \qquad [j^{r+1}]\widehat{Q}_{6,r+1} = \sigma_3. \tag{2.82}$$

On the generic locus these can be factorized as

$$\psi_4(j) = (\sigma_2 + 1 - r)\, \frac{\prod_{A=1}^{r+2}(j - \widehat{b}_{4,A})}{j^2(j-1728)^2 \prod_{I=1}^{r}(j-p_I)}, \tag{2.83}$$

$$\psi_6(j) = \sigma_3\, \frac{\prod_{A=1}^{r+1}(j - \widehat{b}_{6,A})}{j^2(j-1728)^2 \prod_{I=1}^{r}(j-p_I)}. \tag{2.84}$$

The local indicial data impose

$$(\sigma_2 + 1 - r)\, \frac{\prod_{A=1}^{r+2} \widehat{b}_{4,A}}{1728^2 \prod_{I=1}^{r} p_I} = \frac{2}{9}, \tag{2.85}$$

$$(\sigma_2 + 1 - r)\, \frac{\prod_{A=1}^{r+2} (1728 - \widehat{b}_{4,A})}{1728^2 \prod_{I=1}^{r} (1728 - p_I)} = -\frac{1}{4}. \tag{2.86}$$

The second relation is absent in the $\ell = 6r$ case because there ψ_4 has only a simple pole at $j = 1728$.

For completeness, if the coefficient functions in the τ-plane are normalized to have unit leading q-coefficient, the cusp indicial equation gives

$$\ell = 6r: \qquad \mu_2 = r, \qquad \mu_4 = \sigma_2 + \frac{3r-1}{18}, \qquad \mu_6 = -\sigma_3, \tag{2.87}$$

$$\ell = 6r+3: \qquad \mu_2 = r + \frac{1}{2}, \qquad \mu_4 = \sigma_2 + \frac{6r+1}{36}, \qquad \mu_6 = -\sigma_3. \tag{2.88}$$

These follow by expanding

$$\mathcal{D}^{(3)} q^\alpha = \alpha\left(\alpha - \frac{1}{6}\right)\left(\alpha - \frac{1}{3}\right) q^\alpha + O(q^{\alpha+1}), \qquad \mathcal{D}^{(2)} q^\alpha = \alpha\left(\alpha - \frac{1}{6}\right) q^\alpha + O(q^{\alpha+1}) \tag{2.89}$$

and matching the resulting cubic to $\prod_{a=0}^{2}(\alpha - \alpha_a)$.

2.8 No-log conditions at a movable zero

The local exponents $(0, 1, 3)$ follow from the leading indicial polynomial, but they do not by themselves guarantee that $j = p_I$ is an apparent singularity. The logarithms are absent only when two additional compatibility conditions hold.

Set $x = j - p_I$ and expand

$$\begin{aligned} \psi_2 &= -\frac{1}{x} + A_I + O(x), \\ \psi_4 &= \frac{B_I}{x} + \widetilde{B}_I + O(x), \\ \psi_6 &= \frac{C_I}{x} + \widetilde{C}_I + O(x). \end{aligned} \tag{2.90}$$

For either allowed residue class,

$$A_I = \frac{2}{p_I} + \begin{cases} \dfrac{3}{2(p_I - 1728)}, & \ell = 6r, \\ \dfrac{1}{p_I - 1728}, & \ell = 6r+3, \end{cases} - \sum_{J \neq I} \frac{1}{p_I - p_J}. \tag{2.91}$$

Insert the regular series

$$\chi = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots \tag{2.92}$$

into (2.23). At order x^{-1} one finds

$$-2u_2 + B_I u_1 + C_I u_0 = 0, \qquad u_2 = \frac{1}{2}(B_I u_1 + C_I u_0). \tag{2.93}$$

At the resonant order x^0, the coefficient multiplying u_3 vanishes because the third local exponent is 3. The remaining equation is

$$2(A_I + B_I)u_2 + (\widetilde{B}_I + C_I)u_1 + \widetilde{C}_I u_0 = 0. \tag{2.94}$$

Using (2.93), the coefficients of the independent data u_0 and u_1 must vanish separately. The two apparentness conditions are therefore

$$\boxed{(A_I + B_I)B_I + \widetilde{B}_I + C_I = 0,} \tag{2.95}$$

$$\boxed{(A_I + B_I)C_I + \widetilde{C}_I = 0.} \tag{2.96}$$

These equations generally couple the data belonging to different p_I through the finite parts in (2.90); one should therefore not assume without proof that the constraints at different movable zeros are algebraically independent.

There is one additional resonance in the $\ell = 6r+3$ case at $j = 1728$. Writing $x = j-1728$,

$$\begin{aligned}
\psi_2 &= \frac{1}{x} + a_0 + O(x),\\
\psi_4 &= -\frac{1}{4x^2} + \frac{b_{-1}}{x} + O(1),\\
\psi_6 &= \frac{c_{-2}}{x^2} + O(x^{-1}),
\end{aligned} \tag{2.97}$$

the exponent $1/2$ resonates with $3/2$ one order later. The corresponding compatibility condition is

$$\boxed{-\frac{1}{4}a_0 + \frac{1}{2}b_{-1} + c_{-2} = 0.} \tag{2.98}$$

This follows by substituting $\chi = x^{1/2}(u_0 + u_1 x + \cdots)$ and setting the coefficient at the resonant order to zero. For $\ell = 6r$, the exponents at this point are $(0, 1/2, 1)$. The apparent $0 \to 1$ resonance is automatically satisfied because both ψ_4 and ψ_6 have only simple poles at $j = 1728$; hence there is no additional elliptic no-log equation in that residue class.

2.9 Why an irreducible three-character system has $\ell = 3m$

We now prove the missing divisibility result. Let ϱ be the three-dimensional modular representation carried by the characters [2] and set

$$\mathsf{S} := \varrho(S), \qquad \mathsf{R} := \varrho(ST), \qquad \mathsf{T} := \varrho(T). \tag{2.99}$$

Let

$$\mathbf{F}(\tau) := \begin{pmatrix} \chi_0(\tau) \\ \chi_1(\tau) \\ \chi_2(\tau) \end{pmatrix}.$$

Its weight-zero modular transformation law is

$$\mathbf{F}(\gamma\tau) = \varrho(\gamma)\mathbf{F}(\tau), \qquad \gamma \in SL(2,\mathbb{Z}).$$

Since the central element $-I_2$ fixes every point of the upper half-plane, $(-I_2)\tau = \tau$, we have, for every τ,

$$\mathbf{F}(\tau) = \mathbf{F}((-I_2)\tau) = \varrho(-I_2)\mathbf{F}(\tau).$$

Thus each row of $\varrho(-I_2)-\mathbf{1}_3$ gives a constant linear relation among the three component functions χ_0,χ_1,χ_2. Their linear independence therefore implies

$$\varrho(-I_2)=\mathbf{1}_3.$$

Hence $-I_2$ lies in the kernel of ϱ, and the representation descends to

$$SL(2,\mathbb{Z})/\{\pm I_2\}=PSL(2,\mathbb{Z}).$$

Since the standard generators obey

$$S^2=(ST)^3=-I_2$$

in $SL(2,\mathbb{Z})$, the matrices defined in (2.99) satisfy

$$\mathsf{S}^2=\mathbf{1}_3,\qquad \mathsf{R}^3=\mathbf{1}_3. \tag{2.100}$$

Over $\mathbb{C}$, the polynomials x^2-1 and x^3-1 have no repeated roots. Hence both S and R are diagonalizable.

Suppose first that S were scalar. Then the representation would be generated, up to a scalar, by the single diagonalizable matrix R and would have invariant eigenlines. It would be reducible. Thus in an irreducible three-dimensional representation, S is non-scalar. Its eigenvalues lie in $\{1,-1\}$, so one of its eigenspaces has dimension two.

If R were scalar, reducibility would again be immediate. If it were non-scalar but had a repeated eigenvalue, one of its eigenspaces would also have dimension two. Any two two-dimensional subspaces of a three-dimensional vector space have a nonzero intersection. A nonzero vector in the intersection would be a simultaneous eigenvector of S and R; its span would be invariant under the two generators of $PSL(2,\mathbb{Z})$. This contradicts irreducibility. Therefore the three eigenvalues of R are distinct:

$$\boxed{\operatorname{Spec}(\mathsf{R})=\{1,\omega,\omega^2\},\qquad \omega=e^{2\pi i/3}.} \tag{2.101}$$

In particular,

$$\det\mathsf{R}=1. \tag{2.102}$$

Since $\mathsf{R}=\mathsf{ST}$ and $\mathsf{S}^{-1}=\mathsf{S}$, we have

$$\mathsf{T}=\mathsf{SR},\qquad \det\mathsf{T}=\det\mathsf{S}\det\mathsf{R}=\det\mathsf{S}=\pm1. \tag{2.103}$$

In the chosen non-logarithmic character basis, the cusp expansions imply

$$\mathsf{T}=\operatorname{diag}\big(e^{2\pi i\alpha_0},e^{2\pi i\alpha_1},e^{2\pi i\alpha_2}\big),\qquad \det\mathsf{T}=e^{2\pi i(\alpha_0+\alpha_1+\alpha_2)}.$$

$$\mathsf{T}=\operatorname{diag}\big(e^{2\pi i\alpha_0},e^{2\pi i\alpha_1},e^{2\pi i\alpha_2}\big),\qquad \det\mathsf{T}=e^{2\pi i(\alpha_0+\alpha_1+\alpha_2)}. \tag{2.104}$$

For three characters the valence formula is

$$\alpha_0+\alpha_1+\alpha_2=\frac{3(3-1)}{12}-\frac{\ell}{6}=\frac{1}{2}-\frac{\ell}{6}. \tag{2.105}$$

Squaring (2.104) and using (2.103),

$$1 = (\det \mathsf{T})^2 = \exp\left[4\pi i\left(\frac{1}{2} - \frac{\ell}{6}\right)\right]$$
$$= \exp\left(-\frac{2\pi i \ell}{3}\right). \tag{2.106}$$

Since the Wronskian index ℓ is non-negative by definition, (2.106) implies

$$\boxed{\ell = 3m, \qquad m \in \mathbb{Z}_{\geq 0}.} \tag{2.107}$$

There is an instructive local version of the same proof. The local order-three monodromy at ρ is represented by R up to inversion, depending on the orientation chosen for the local loop; this does not change the spectrum in (2.101). After choosing an eigenbasis, holomorphy in the uniformizing coordinate $\tau - \rho$, together with $j \sim (\tau - \rho)^3$, gives

$$\lambda_0^{(\rho)} = n_0, \qquad \lambda_1^{(\rho)} = n_1 + \frac{1}{3}, \qquad \lambda_2^{(\rho)} = n_2 + \frac{2}{3}, \qquad n_a \in \mathbb{Z}_{\geq 0}. \tag{2.108}$$

Equation (2.38) then gives

$$\frac{\ell_\rho}{6} = \lambda_0^{(\rho)} + \lambda_1^{(\rho)} + \lambda_2^{(\rho)} - 1 = n_0 + n_1 + n_2. \tag{2.109}$$

Thus ℓ_ρ is a multiple of six. If ν_i denotes vanishing order in the uniformizing coordinate $\tau - i$, a zero at the order-two elliptic point contributes $3\nu_i$ to ℓ. If ν_p denotes vanishing order in the ordinary coordinate $j - p$, a generic zero contributes $6\nu_p$. Therefore

$$\ell = \ell_\rho + \ell_i + \sum_p \ell_p \in 3\mathbb{Z}, \tag{2.110}$$

which reproduces (2.107) directly in the local Wronskian language.

For comparison with the six residue classes $\ell = 6r + u$, the minimal fixed elliptic contributions are

u	(ℓ_ρ, ℓ_i)
0	$(0,0)$
1	$(4,3)$
2	$(2,0)$
3	$(0,3)$
4	$(4,0)$
5	$(2,3)$

The four classes $u = 1, 2, 4, 5$ have $\ell_\rho \notin 6\mathbb{Z}$ and are therefore incompatible with irreducibility. The surviving cases are exactly $u = 0, 3$, as stated in (2.1). In the $u = 1$ row the displayed fixed contribution already equals seven, so that row starts at $\ell = 7$ (equivalently $r \geq 1$). The isolated value $\ell = 1$ is absent; in the modular-form construction this is the familiar absence of a nonzero holomorphic modular form of weight two for $PSL(2, \mathbb{Z})$.

More generally, the coefficient of ∂_j^2 can be written directly in terms of the local Wronskian contributions as

$$\boxed{\psi_2(j) = \frac{2 - \ell_\rho/6}{j} + \frac{3/2 - \ell_i/6}{j - 1728} - \sum_I \frac{\nu_I}{j - p_I},} \tag{2.111}$$

where ν_I is the order of the Wronskian zero at the generic point p_I.

2.10 Reducible exceptions and the precise physical statement

The proof above also explains why the known $[3,2]$ and $[3,4]$ solutions obtained in [3] and [4] respectively, do not contradict (2.107). Let a two-character system have exponents α_0, α_1 and Wronskian index $\ell_{(2)}$. Its valence formula is

$$\alpha_0 + \alpha_1 = \frac{1}{6} - \frac{\ell_{(2)}}{6}. \tag{2.112}$$

Adjoin a constant solution, whose exponent is zero. If the resulting three-component vector has index $\ell_{(3)}$, then

$$\alpha_0 + \alpha_1 + 0 = \frac{1}{2} - \frac{\ell_{(3)}}{6}. \tag{2.113}$$

Equating (2.112) and (2.113) gives

$$\boxed{\ell_{(3)} = \ell_{(2)} + 2.} \tag{2.114}$$

Thus

$$\mathbf{1} \oplus (2,0) \quad \text{has} \quad \ell = 2, \qquad \mathbf{1} \oplus (2,2) \quad \text{has} \quad \ell = 4. \tag{2.115}$$

In each case the constant character spans an invariant one-dimensional subspace, so the modular representation is reducible.

This also sharpens a common phrase used in the second-order argument. If the zeroth-order coefficient of a second-order equation vanishes, the solution space does not literally become one-dimensional. Rather, the operator factorizes as

$$\partial_j^2 + P(j)\partial_j = \big(\partial_j + P(j)\big)\partial_j, \tag{2.116}$$

so a constant solution appears and spans an invariant line. The correct obstruction is reducibility of the modular representation.

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