COEP TECHNOLOGICAL UNIVERSITY ·PUNE

SUMMER INTERNSHIP REPORT

MODULAR FORMS IN PHYSICS

Presented by:
Yash C. Patki

MIS No :
612410195

Supervisor:
Dr. Arpit Das

COEP Mentor:
Dr. V.V. Palande

Session Year
2025-26 (II Year)

Contents

List of Symbols and Abbreviations **iii**

1 Introduction to The Modular Group **2**
1.1 The Modular Group: Description and Introduction 2
1.1.1 Generators of the Group 2
1.1.2 The Action on the Upper-Half Plane 3
1.1.3 Definition of a Modular Form of Weight k 3
1.1.4 The Vanishing of Odd-Weight Modular Forms 3
1.1.5 Definition of a Modular Function 4

2 Project Background **5**
2.1 Engineering Applications: Materials and Periodic Media 5
2.1.1 Crystallography and Mechanical Stress in 2D Materials 5
2.1.2 Acoustic Metamaterials and Phononic Crystals 5

3 Eisenstein Series and a Basis **7**
3.1 Definition and Construction 7
3.1.1 Transformation Behavior under $\mathrm{SL}(2,\mathbb{Z})$ 7
3.2 The Normalized Eisenstein Series and q-Expansion 8
3.3 Generating a Basis for Modular Forms 9

4 The Fundamental Domain of the Modular Group **10**
4.1 Geometric Definition and Boundaries 10
4.2 Visualizing the Geometric Tiling 11
4.3 Physical Role in Field Theories 11

5 The Valence Formula **12**
5.1 The Theorem and Mathematical Statement 12
5.2 Physical and Design Implications 13

6 The Quasi-modular Eisenstein Series E_2 and the Ramanujan-Serre Derivative **14**
6.1 The Non-Modularity of the Weight-2 Eisenstein Series 14
6.2 The Breakdown of Standard Differentiation 15
6.3 The Ramanujan-Serre Differential Operator 15
6.3.1 Proof of Covariant Modularity 16
6.4 Ramanujan's Identities between Eisenstein Series 16
6.4.1 Derivation of $\theta E_2 = \frac{E_2^2 - E_4}{12}$ 16
6.4.2 Derivation of $\theta E_4 = \frac{E_2 E_4 - E_6}{3}$ 18
6.4.3 Derivation of $\theta E_6 = \frac{E_2 E_6 - E_4^2}{2}$ 18

7 Modular Linear Differential Equations (MLDE) **19**
7.1 General Formulation and Definition 19
7.1.1 The Monic Constraint and Base Forms 20
7.2 Detailed Analysis of the Second-Order MLDE 20
7.3 Solving via the Frobenius Method (q-Expansion) 21
7.3.1 Explicit Derivation of the Coefficients and Recursion Relations . . 21
7.3.2 The First Two Higher-Order Recursion Terms 22
7.3.3 Physical Admissibility Requirements 23
7.3.4 Mapping Recursion Parameters against Valence Formula Bounds . 23
7.3.5 Geometric Elimination of Coefficient Divergences 24

Bibliography **25**

List of Symbols and Abbreviations

The following list describes the primary mathematical symbols, operators, and abbreviations utilized throughout this report:

Mathematical Spaces and Groups

$\mathbb{H}$	Upper half of the complex plane, $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$
$\mathbb{C}$	The set of complex numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_{\geq 0}$	The set of non-negative integers
$\mathrm{SL}(2, \mathbb{Z})$	Special Linear Group of 2×2 matrices with integer entries and determinant 1
$\mathrm{PSL}(2, \mathbb{Z})$	Projective Modular Group, $\mathrm{SL}(2, \mathbb{Z})/\{\pm I\}$
$\mathcal{F}$	Standard fundamental domain of the modular group
M_k	Vector space of holomorphic modular forms of weight k

Special Functions and Series

$G_k(\tau)$	Unnormalized Eisenstein series of weight k
$E_k(\tau)$	Normalized Eisenstein series of weight k (leading Fourier coefficient scaled to 1)
$\zeta(s)$	Riemann Zeta Function
B_k	The k-th Bernoulli number
$\sigma_{k-1}(n)$	Divisor function summing the $(k-1)$-th powers of the divisors of n
$\Delta(\tau)$	The modular discriminant function of weight 12
$\tau(n)$	Ramanujan's tau function (Fourier coefficients of Δ)
q	The modular parameter, defined as $e^{2\pi i \tau}$

Operators and Differential Notation

γ	A general transformation matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ inside $\mathrm{SL}(2,\mathbb{Z})$
S, T	The fundamental generator matrices of the modular group
$\vert_k$	The modular slash action of weight k
$\mathrm{ord}_z(f)$	Order of vanishing (multiplicity of a zero or pole) of a function f at point z
θ	The Ramanujan fundamental differential operator, $q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$
$\mathcal{D}_k$	The modular covariant Ramanujan-Serre derivative operator

MLDE and Frobenius System Parameters

h, h_1, h_2	Indicial roots / conformal weights of the system boundary state character
c_m	General Fourier expansion coefficients computed via the Frobenius method
μ	Free independent parameter parameterizing the second-order Serre MLDE
Λ	Collected parameter shortcut defined as $\mu + \frac{k}{144}$

Core Abbreviations

MLDE	Modular Linear Differential Equation
CFT	Conformal Field Theory
RCFT	Rational Conformal Field Theory
COEP	College of Engineering Pune (COEP Technological University)
MIS	Management Information System (Student Identification Number)
UV	Ultraviolet (used in reference to high-energy coordinate divergences)

Chapter 1

Introduction to The Modular Group

1.1 The Modular Group: Description and Introduction

In mathematical physics—particularly in conformal field theory and string theory—the geometric symmetries of a system are often governed by the **Modular Group**, denoted as $\mathrm{SL}(2,\mathbb{Z})$. This group consists of all 2×2 matrices with integer entries that have a determinant equal to 1:

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \right\} \tag{1.1}$$

The modular group acts naturally on the upper half of the complex plane, defined as $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. For any matrix $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, the action is given by the linear fractional transformation:

$$\gamma \cdot \tau = \frac{a\tau + b}{c\tau + d} \tag{1.2}$$

Because the matrices γ and $-\gamma$ yield the exact same transformation on $\mathbb{H}$, we often work with the **Projective Modular Group**, defined as $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$.

1.1.1 Generators of the Group

The entire modular group can be generated by compounding just two fundamental operations, corresponding to the matrices S and T:

- **Inversion (S):** Maps $\tau \to -1/\tau$, swapping large-scale and small-scale geometries.

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.3}$$

- **Translation (T):** Maps $\tau \to \tau + 1$, shifting the complex coordinate by a single unit.

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1.4}$$

These generators obey the relations $S^2 = (ST)^3 = -I$. In physical applications, such as the study of torus worldsheets in string theory, these transformations ensure that the physical observables remain invariant under global reparameterizations of the geometry.

1.1.2 The Action on the Upper-Half Plane

The geometric significance of the modular group is realized through its conformal action on the upper-half plane $\mathbb{H}$. For any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, the imaginary part of the transformed variable $\tau' = \gamma \cdot \tau$ scales according to the relation:

$$\mathrm{Im}(\gamma \cdot \tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \tag{1.5}$$

This relation is critical for establishing bounded physical behaviors. Since $\mathrm{Im}(\tau) > 0$, it guarantees that the transformed parameter $\gamma \cdot \tau$ always remains strictly within the upper half-plane.

1.1.3 Definition of a Modular Form of Weight k

More generally, we define a holomorphic function $f(\tau)$ on the upper half-plane $\mathbb{H}$ to be a **Modular Form of weight** k (where $k \in \mathbb{Z}$) under the full modular group $\mathrm{SL}(2, \mathbb{Z})$ if it satisfies the following functional transformation law for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \tag{1.6}$$

Additionally, the function must remain holomorphic as $\tau \to i\infty$ (meaning its Fourier q-expansion contains no negative powers of q). When $k = 0$, this definition naturally reduces to the standard invariant modular function described below.

1.1.4 The Vanishing of Odd-Weight Modular Forms

An important algebraic constraint emerges immediately from this definition when considering odd integer weights. The negative identity matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is a valid element of $\mathrm{SL}(2, \mathbb{Z})$ since its determinant is $(-1)(-1) - (0)(0) = 1$.

Evaluating the modular transformation rule specifically for this matrix yields:

$$f\left(\frac{-1\cdot\tau+0}{0\cdot\tau-1}\right)=(0\cdot\tau-1)^k f(\tau) \tag{1.7}$$

Simplifying both sides of the equation gives:

$$f(\tau)=(-1)^k f(\tau) \tag{1.8}$$

If the weight k is an "odd integer", then $(-1)^k=-1$, which reduces the expression to:

$$f(\tau)=-f(\tau)\implies 2f(\tau)=0\implies f(\tau)=0 \tag{1.9}$$

This mathematically proves that any modular form of an odd weight k under $\mathrm{SL}(2,\mathbb{Z})$ is trivially zero. Consequently, non-trivial holomorphic modular forms can only exist for even weights, a foundational constraint that structures our entire study of physical partition bases.

1.1.5 Definition of a Modular Function

A meromorphic function $f(\tau)$ on $\mathbb{H}$ is classified as a weight-0 modular form (commonly known as a **Modular Function**) if it is completely invariant under all transformations of the modular group. Formally, for all $\gamma\in\mathrm{SL}(2,\mathbb{Z})$, it must satisfy:

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=f(\tau) \tag{1.10}$$

Because the transformation matrix $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ maps $\tau\to\tau+1$, any valid modular function is inherently periodic with a period of 1:

$$f(\tau+1)=f(\tau) \tag{1.11}$$

This exact translational periodicity allows us to expand $f(\tau)$ as a Fourier series in terms of the parameter $q=e^{2\pi i\tau}$, traditionally referred to as the q**-expansion** of the function:

$$f(\tau)=\sum_{n=-\infty}^{\infty}c(n)q^n \tag{1.12}$$

In mechanical lattice frameworks and thermodynamic field profiles, the coefficients $c(n)$ directly enumerate physical states, discrete vibrational frequencies, or structural degeneracy levels within periodic structural media.

Chapter 2

Project Background

2.1 Engineering Applications: Materials and Periodic Media

While modular forms originate in pure mathematics and mathematical physics, their underlying group structures ($\mathrm{SL}(2,\mathbb{Z})$) directly govern physical properties critical to mechanical engineering disciplines, particularly in advanced materials design, vibration isolation, and structural mechanics.

2.1.1 Crystallography and Mechanical Stress in 2D Materials

In the design of micro-electromechanical systems (MEMS) and advanced two-dimensional materials (like graphene or transition metal dichalcogenides), mechanical engineers must model the elastic strain and thermal conductivity of atomic lattices.

A perfect 2D hexagonal or square crystal lattice can be mapped mathematically to a torus using a complex parameter $\tau \in \mathbb{H}$. The mechanical properties of the crystal—such as its tensor invariants, shear modulus, and stress-strain relationships—must remain invariant regardless of how the underlying lattice base vectors are chosen. Because switching between equivalent lattice bases is governed precisely by the modular transformations:

$$\tau \to \tau + 1 \quad \text{and} \quad \tau \to -\frac{1}{\tau} \tag{2.1}$$

the energetic partition functions that predict macroscopic mechanical failure, buckling points, and thermal expansion coefficients are modeled using modular forms.

2.1.2 Acoustic Metamaterials and Phononic Crystals

Modern mechanical engineering heavily focuses on **acoustic metamaterials**—engineered structures designed to manipulate, redirect, or completely block mechanical vibrations

and sound waves. These structures rely on a periodic arrangement of mass-spring resonators or varying geometric densities.

When modeling wave propagation through these periodic structures, engineers analyze the dispersion relations within the Brillouin zone. For structures with specific underlying geometric symmetries (like a periodic lattice of acoustic ducts or vibrating membranes), the wave equations can be mapped to the upper-half plane. Modular forms provide an exact analytical tool to calculate:

- **Phononic Bandgaps:** Predicting the exact frequency ranges where structural vibrations are completely canceled out.
- **Acoustic Invisibility Cloaks:** Mapping the continuous deformation of mechanical fields around an object using conformal transformations that respect modular symmetries.

By embedding modular symmetries into structural design algorithms, mechanical engineers can optimize topological insulators that guide shockwaves away from critical aerospace or structural components.

Chapter 3

Eisenstein Series and a Basis

To systematically construct explicit examples of modular forms of higher weights, we look to convergent infinite sums over lattice coordinates. The most fundamental and structurally vital examples of these are the **Eisenstein Series**. These series serve as a foundational basis for the entire vector space of modular forms.

3.1 Definition and Construction

For an even integer $k > 2$, the Eisenstein series of weight k is defined by summing over all non-zero integer pairs (m, n) that form a two-dimensional lattice grid. For a parameter $\tau \in \mathbb{H}$, the series is expressed as:

$$G_k(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau + n)^k} \tag{3.1}$$

Because $k > 2$ and is strictly even, this series converges absolutely and uniformly on any compact subset of the upper half-plane.

3.1.1 Transformation Behavior under $\mathbf{SL}(2, \mathbb{Z})$

By applying the generator matrices of the modular group, we can observe how $G_k(\tau)$ transforms. Under the translation operator T ($\tau \to \tau + 1$), the lattice structure remains unchanged, meaning:

$$G_k(\tau + 1) = G_k(\tau) \tag{3.2}$$

Under the inversion operator S ($\tau \to -1/\tau$), evaluating the summation yields a systematic scaling factor dependent on the weight k:

$$G_k\left(-\frac{1}{\tau}\right) = \tau^k G_k(\tau) \tag{3.3}$$

Generally, for any transformation matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, the Eisenstein series satisfies the functional equation:

$$G_k\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k G_k(\tau) \tag{3.4}$$

This confirms that $G_k(\tau)$ is a valid holomorphic modular form of weight k.

3.2 The Normalized Eisenstein Series and q-Expansion

In both physical and engineering applications, it is standard practice to normalize these series so that the constant leading term of their Fourier series expansion is exactly equal to 1. The normalized Eisenstein series are denoted as $E_k(\tau)$ and are scaled using Riemann zeta function values:

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \tag{3.5}$$

The term $\zeta(k)$ appearing in the normalization denominator is the **Riemann Zeta Function**. For any complex variable s with $\mathrm{Re}(s) > 1$, it is defined by the absolutely convergent infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.6}$$

Because the Eisenstein series requires an even weight $k > 2$, the values of $\zeta(k)$ evaluate cleanly to specific circular fractions involving powers of π, as proven by Euler's solution to the Basel problem:

$$\zeta(k) = \frac{(-1)^{\frac{k}{2}+1} B_k (2\pi)^k}{2(k!)} \tag{3.7}$$

where B_k represents the k-th Bernoulli number. In Equation (3.5), $\sigma_{k-1}(n)$ is the divisor function summing the $(k-1)$-th powers of the positive divisors of n, and $q = e^{2\pi i \tau}$.

where B_k represents the k-th Bernoulli number, $q = e^{2\pi i \tau}$, and $\sigma_{k-1}(n)$ is the divisor function summing the $(k-1)$-th powers of the divisors of n. The two most vital low-weight instances are $E_4(\tau)$ and $E_6(\tau)$:

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \tag{3.8}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \tag{3.9}$$

3.3 Generating a Basis for Modular Forms

A remarkable algebraic property of the vector space of modular forms, denoted as $M_k(\mathrm{SL}(2,\mathbb{Z}))$, is that it is finite-dimensional. Furthermore, the entire graded ring of modular forms is generated exclusively by the polynomials of just these two foundational series: E_4 and E_6.

This means that any valid modular form $f(\tau)$ of any arbitrary weight k can be uniquely expressed as a linear combination of products of E_4 and E_6:

$$f(\tau) = \sum_{4a+6b=k} c_{a,b} \left[E_4(\tau)\right]^a \left[E_6(\tau)\right]^b \tag{3.10}$$

where $a, b \in \mathbb{Z}_{\geq 0}$ are non-negative integers, and the coefficients $c_{a,b} \in \mathbb{C}$ the set of complex numbers. For any physical or mechanical framework whose boundary conditions exhibit modular symmetry, this finite-dimensional basis provides a massive simplification. Instead of needing an infinite number of independent functions to track state configurations, the entire physical state space can be completely mapped using explicit polynomial combinations of just two fundamental building blocks: the weight-4 and weight-6 Eisenstein series.

Chapter 4

The Fundamental Domain of the Modular Group

Because the transformations of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ tile the entire upper half-plane $\mathbb{H}$ with repeating symmetric regions, tracking functions across the entire plane presents computational redundancies. To eliminate these redundancies, we restrict our geometric analysis to a single, non-repeating master segment known as the **Fundamental Domain**, denoted as $\mathcal{F}$.

4.1 Geometric Definition and Boundaries

The standard fundamental domain for the modular group is defined as the open region bounded by the vertical lines $\mathrm{Re}(\tau) = \pm 1/2$ and bounded below by the unit circle arc $|\tau| = 1$. Mathematically, this set is defined as:

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} \,\middle|\, -\frac{1}{2} < \mathrm{Re}(\tau) < \frac{1}{2} \text{ and } |\tau| > 1 \right\} \tag{4.1}$$

To ensure that every single orbit under the action of the group maps to exactly one point in the closure, we include half of the bounding edges explicitly:

$$\overline{\mathcal{F}} = \mathcal{F} \cup \left\{ \tau \in \mathbb{H} \,\middle|\, \mathrm{Re}(\tau) = -\frac{1}{2}, |\tau| \geq 1 \right\} \cup \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| = 1, -\frac{1}{2} \leq \mathrm{Re}(\tau) \leq 0 \right\} \tag{4.2}$$

Every coordinate position in the upper half-plane can be mapped directly into this strip via a finite sequence of compounding operations using the generator matrices S and T.

4.2 Visualizing the Geometric Tiling

When the generator transformations $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ are applied repeatedly to the edges of the master domain $\mathcal{F}$, they generate an infinite array of non-overlapping geodesic triangles that pack the upper half-plane. This mapping pattern is visualized in Figure 4.1.

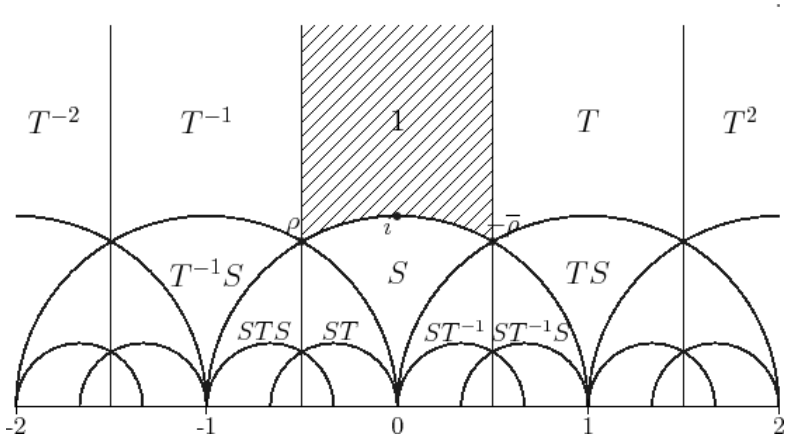


Figure 4.1: The standard fundamental domain $\mathcal{F}$ of $\mathrm{SL}(2,\mathbb{Z})$ in the upper half-plane, showing adjacent modular tiles generated via S and T actions.

4.3 Physical Role in Field Theories

In modern physics, constraining coordinates to the fundamental domain is a necessary mechanism to prevent mathematical blow-ups:

- **String Theory Worldsheets:** When calculating closed-string loop interactions, the topology of the trace forms a continuous torus. Integrating parameters across all of $\mathbb{H}$ would sum identical tori geometric frames an infinite number of times, causing physical observable amplitudes to diverge. Restricting integration limits cleanly to $\mathcal{F}$ provides an elegant UV cutoff.
- **Topological Mechanical Lattices:** In advanced mechanical structural materials, computing acoustic wave bandgaps across an expansive repeating grid is computationally inefficient. The fundamental domain acts as an irreducible structural zone, allowing engineers to map full vibrational isolation capacities from a minimal geometric envelope.

Chapter 5

The Valence Formula

A foundational result in the theory of modular forms under the group $\mathrm{SL}(2,\mathbb{Z})$ is the **Valence Formula**. This formula acts as a rigid geometric and analytic conservation law, linking the number of zeros and poles a modular form has inside the closure of the fundamental domain directly to its algebraic weight k. It is widely referred to as the $\frac{k}{12}$ **Theorem** because when it is applied to the fundamental modular discriminant $\Delta(\tau)$—which has a weight of $k = 12$ and vanishes uniquely with an order of 1 at the cusp (∞)—the weighted sum of coordinates evaluates cleanly to exactly $12/12 = 1$.

5.1 The Theorem and Mathematical Statement

For any non-zero meromorphic modular form $f(\tau)$ of weight k, the weighted sum of its orders of vanishing across the upper half-plane is always equal to exactly one-twelfth of its weight:

$$\mathrm{ord}_\infty(f) + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \sum_{z\in\mathcal{F}, z\neq i,\rho} \mathrm{ord}_z(f) = \frac{k}{12} \tag{5.1}$$

where:

- $\mathrm{ord}_z(f)$ is the standard order of vanishing (the multiplicity of the zero or pole) at a point z.
- $\mathrm{ord}_\infty(f)$ is the order of vanishing at the cusp $(q \to 0)$.
- i and $\rho = e^{2\pi i/3}$ are the special elliptic fixed points on the boundary of the fundamental domain. They are scaled by weights $1/2$ and $1/3$ due to their localized geometric symmetries under the operations S and ST.

5.2 Physical and Design Implications

In both mathematical physics models and mechanical periodic structural frameworks, this conservation law forces rigid constraints on system properties:

- **Vector Space Boundaries:** The formula establishes the exact finite dimensions of the spaces of modular forms. For instance, because the left side must be non-negative for regular holomorphic forms, it proves why there are zero non-trivial forms for negative weights, odd weights, or for the specific weight $k = 2$.
- **Tracking Field Nodes:** In lattice engineering and acoustic metamaterials, the zeros of the function represent physical nodes where vibrational displacement drops completely to zero. The valence formula allows engineers to predict the exact number of these quiet spots solely from the structural weight parameter k, removing the need for trial-and-error simulation profiles.

Chapter 6

The Quasi-modular Eisenstein Series E_2 and the Ramanujan-Serre Derivative

When studying the vector spaces of modular forms, an operational barrier arises when attempting to differentiate a holomorphic modular form. Standard differentiation destroys modularity. To restore this symmetry, we must utilize the quasi-modular Eisenstein series $E_2(\tau)$ to construct a covariant differential operator known as the **Ramanujan-Serre Derivative**.

6.1 The Non-Modularity of the Weight-2 Eisenstein Series

The weight-2 Eisenstein series $G_2(\tau)$ is a well-defined holomorphic function on the upper half-plane $\mathbb{H}$, provided its lattice summation is carried out in a specific, conditional order. However, because this series does not converge absolutely, rearranging the terms changes the value of the sum. It is precisely this lack of absolute convergence that breaks full modular symmetry under coordinate inversion.

To evaluate its properties explicitly, we define the normalized weight-2 Eisenstein series $E_2(\tau)$ via its conditionally convergent Fourier q-expansion:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \tag{6.1}$$

where $q = e^{2\pi i \tau}$ and $\sigma_1(n)$ is the sum of divisors of n.

While $E_2(\tau)$ is invariant under translation ($E_2(\tau + 1) = E_2(\tau)$), its lack of absolute

convergence introduces a non-vanishing additive error term under inversion S:

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6i\tau}{\pi} \tag{6.2}$$

More generally, for any transformation matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, the full transformation law is given by:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6ic(c\tau+d)}{\pi} \tag{6.3}$$

Functions that transform according to this relation are classified as **Quasi-Modular Forms** of weight 2. Though $E_2(\tau)$ fails to be a true modular form, its term is exactly what allows us to correct the non-modular behavior of standard derivatives.

6.2 The Breakdown of Standard Differentiation

Let $f(\tau)$ be a holomorphic modular form of weight k. Applying the standard derivative operator $\frac{d}{d\tau}$ to its modular transformation law:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \tag{6.4}$$

and applying the chain rule yields:

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) \cdot \frac{1}{(c\tau+d)^2} = kc(c\tau+d)^{k-1} f(\tau) + (c\tau+d)^k f'(\tau) \tag{6.5}$$

Multiplying through by $(c\tau+d)^2$ reveals that the derivative contains an unwanted term:

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2} f'(\tau) + kc(c\tau+d)^{k+1} f(\tau) \tag{6.6}$$

Because of the second term on the right, $f'(\tau)$ is fundamentally "not" a modular form of weight $k+2$.

6.3 The Ramanujan-Serre Differential Operator

To cancel out this unwanted $kc(c\tau+d)^{k+1} f(\tau)$ error term, we leverage the anomalous additive term from $E_2(\tau)$. We define the "Ramanujan-Serre Derivative" (or modular derivative), denoted as $\mathcal{D}_k$, by:

$$\mathcal{D}_k f(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau} f(\tau) - \frac{k}{12} E_2(\tau) f(\tau) \tag{6.7}$$

Expressing this in terms of the differential operator $\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$, the operator simplifies to:

$$\mathcal{D}_k f = \theta(f) - \frac{k}{12} E_2 f \tag{6.8}$$

6.3.1 Proof of Covariant Modularity

We verify that if f is a modular form of weight k, then $\mathcal{D}_k f$ is a true modular form of weight $k+2$. Evaluating the transformation of the two separate parts under $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ gives:

$$\theta(f)\left(\gamma\tau\right) = (c\tau+d)^{k+2}\theta(f)(\tau) + \frac{kc}{2\pi i}(c\tau+d)^{k+1} f(\tau) \tag{6.9}$$

$$\begin{aligned}\left(-\frac{k}{12}E_2 f\right)(\gamma\tau) &= -\frac{k}{12}\left[(c\tau+d)^2 E_2(\tau) + \frac{6ic(c\tau+d)}{\pi}\right]\cdot\left[(c\tau+d)^k f(\tau)\right] \\ &= -(c\tau+d)^{k+2}\frac{k}{12}E_2(\tau)f(\tau) - \frac{kc}{2\pi i}(c\tau+d)^{k+1} f(\tau)\end{aligned} \tag{6.10}$$

Adding equations (6.8) and (6.9) together leaves:

$$(\mathcal{D}_k f)(\gamma\tau) = (c\tau+d)^{k+2}(\mathcal{D}_k f)(\tau) \tag{6.11}$$

This mathematically proves that $\mathcal{D}_k : M_k \to M_{k+2}$ is a perfectly covariant mapping that maps true modular forms of weight k to true modular forms of weight $k+2$.

6.4 Ramanujan's Identities between Eisenstein Series

The proof of Ramanujan's differential identities relies on the finite-dimensionality of the vector spaces of modular forms $M_k(\mathrm{SL}(2,\mathbb{Z}))$ alongside the exact mapping behavior of the Ramanujan-Serre derivative operator $\mathcal{D}_k : M_k \to M_{k+2}$.

6.4.1 Derivation of $\theta E_2 = \frac{E_2^2 - E_4}{12}$

To derive the first identity, we look at the anomalous transformation law of $E_2(\tau)$ under the inversion matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6i\tau}{\pi} \tag{6.12}$$

Differentiating both sides of this equation with respect to τ using the standard chain rule on the left side yields:

$$E_2'\left(-\frac{1}{\tau}\right)\cdot\frac{1}{\tau^2} = 2\tau E_2(\tau) + \tau^2 E_2'(\tau) + \frac{6i}{\pi} \tag{6.13}$$

Multiplying the entire equation through by τ^2 gives:

$$E_2'\left(-\frac{1}{\tau}\right) = \tau^4 E_2'(\tau) + 2\tau^3 E_2(\tau) + \frac{6i\tau^2}{\pi} \tag{6.14}$$

Expressing this in terms of the standard operator $\theta = \frac{1}{2\pi i}\frac{d}{d\tau}$, we multiply through by $\frac{1}{2\pi i}$ to obtain:

$$\theta E_2\left(-\frac{1}{\tau}\right) = \tau^4\theta E_2(\tau) + \frac{\tau^3}{\pi i}E_2(\tau) - \frac{3\tau^2}{\pi^2} \tag{6.15}$$

Now, we construct a helper function $F(\tau) = 12\theta E_2(\tau) - E_2^2(\tau)$. Evaluating $F(-1/\tau)$ using the transformation laws gives:

$$\begin{aligned} F\left(-\frac{1}{\tau}\right) &= 12\left[\tau^4\theta E_2(\tau) + \frac{\tau^3}{\pi i}E_2(\tau) - \frac{3\tau^2}{\pi^2}\right] - \left[\tau^2 E_2(\tau) + \frac{6i\tau}{\pi}\right]^2 \\ &= 12\tau^4\theta E_2(\tau) + \frac{12\tau^3}{\pi i}E_2(\tau) - \frac{36\tau^2}{\pi^2} - \left[\tau^4 E_2^2(\tau) + \frac{12i\tau^3}{\pi}E_2(\tau) - \frac{36\tau^2}{\pi^2}\right] \end{aligned} \tag{6.16}$$

Because $\frac{12}{\pi i} = -\frac{12i}{\pi}$, the matching terms involving $E_2(\tau)$ cancel out exactly, and the constant π^2 terms also cancel, leaving:

$$F\left(-\frac{1}{\tau}\right) = \tau^4\left[12\theta E_2(\tau) - E_2^2(\tau)\right] = \tau^4 F(\tau) \tag{6.17}$$

This confirms that $F(\tau)$ is a true, holomorphic modular form of weight 4. Since the space of weight-4 modular forms $M_4(\mathrm{SL}(2,\mathbb{Z}))$ is one-dimensional and spanned entirely by $E_4(\tau)$, $F(\tau)$ must be a direct scalar multiple of $E_4(\tau)$, meaning $F(\tau) = \alpha E_4(\tau)$. Matching the constant leading terms of their Fourier expansions ($q \to 0$, where $\theta E_2 \to 0, E_2 \to 1, E_4 \to 1$) yields:

$$12(0) - (1)^2 = \alpha(1) \implies \alpha = -1 \tag{6.18}$$

Therefore, $12\theta E_2 - E_2^2 = -E_4$, which rearranges cleanly to yield our first target identity:

$$\theta E_2 = \frac{E_2^2 - E_4}{12} \tag{6.19}$$

6.4.2 Derivation of $\theta E_4 = \frac{E_2E_4 - E_6}{3}$

The weight-4 Eisenstein series $E_4(\tau)$ is a true modular form of weight $k = 4$. Applying the Ramanujan-Serre derivative operator $\mathcal{D}_4$ maps it to a true modular form of weight $4 + 2 = 6$:

$$\mathcal{D}_4 E_4 = \theta E_4 - \frac{4}{12} E_2 E_4 = \theta E_4 - \frac{1}{3} E_2 E_4 \in M_6(\mathrm{SL}(2, \mathbb{Z})) \tag{6.20}$$

Since the space of weight-6 modular forms is one-dimensional and spanned entirely by $E_6(\tau)$, $\mathcal{D}_4 E_4$ must be a scalar multiple of $E_6(\tau)$, meaning $\mathcal{D}_4 E_4 = \beta E_6$. Matching the leading Fourier q-expansion coefficients ($q \to 0$, where $\theta E_4 \to 0, E_2 \to 1, E_4 \to 1, E_6 \to 1$) evaluates the boundary balance:

$$0 - \frac{1}{3}(1)(1) = \beta(1) \implies \beta = -\frac{1}{3} \tag{6.21}$$

Substituting $\beta = -1/3$ back into the operator setup gives:

$$\theta E_4 - \frac{1}{3} E_2 E_4 = -\frac{1}{3} E_6 \tag{6.22}$$

Rearranging this algebraic profile immediately isolates the derivative to yield the second target identity:

$$\theta E_4 = \frac{E_2 E_4 - E_6}{3} \tag{6.23}$$

6.4.3 Derivation of $\theta E_6 = \frac{E_2E_6 - E_4^2}{2}$

Similarly, the weight-6 series $E_6(\tau)$ is a true modular form of weight $k = 6$. Applying the derivative operator $\mathcal{D}_6$ maps it to a true modular form of weight $6 + 2 = 8$:

$$\mathcal{D}_6 E_6 = \theta E_6 - \frac{6}{12} E_2 E_6 = \theta E_6 - \frac{1}{2} E_2 E_6 \in M_8(\mathrm{SL}(2, \mathbb{Z})) \tag{6.24}$$

The vector space $M_8(\mathrm{SL}(2, \mathbb{Z}))$ is also one-dimensional, and it is spanned entirely by $E_4^2(\tau)$ (since $E_8 = E_4^2$). Therefore, $\mathcal{D}_6 E_6 = \gamma E_4^2$. Auditing the constant boundary limits as $q \to 0$ determines the scaling value:

$$0 - \frac{1}{2}(1)(1) = \gamma(1)^2 \implies \gamma = -\frac{1}{2} \tag{6.25}$$

Substituting $\gamma = -1/2$ back into the mapping structure gives:

$$\theta E_6 - \frac{1}{2} E_2 E_6 = -\frac{1}{2} E_4^2 \tag{6.26}$$

Isolating the derivative term yields our final target differential identity:

$$\theta E_6 = \frac{E_2 E_6 - E_4^2}{2} \tag{6.27}$$

Chapter 7

Modular Linear Differential Equations (MLDE)

In mathematical physics—specifically within 2D Rational Conformal Field Theory (RCFT) and string compactifications—a central task is classifying the characters $\chi_i(\tau)$ that define the system's partition function [4]. Because these characters must satisfy precise transformation laws under $\mathrm{SL}(2,\mathbb{Z})$, they form the solution space to a specialized class of ordinary differential equations known as **Modular Linear Differential Equations (MLDEs)**.

7.1 General Formulation and Definition

A linear ordinary differential equation defined on the upper half-plane $\mathbb{H}$ is classified as an MLDE of order n and weight k if its entire vector space of solutions is invariant under the **Modular Slash Action** of $\mathrm{SL}(2,\mathbb{Z})$ [5]. For any integer weight k, the slash operator $|_k$ acts on a holomorphic function $f(\tau)$ for any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ according to the following definition:

$$\left(f|_k\gamma\right)(\tau) = (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) \tag{7.1}$$

Under this notation, a true modular form of weight k is simply a holomorphic function that is completely invariant under this action, satisfying $f|_k\gamma = f$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Using the Ramanujan-Serre derivative $\mathcal{D}_k$ defined in Chapter 6, the standard monic Serre form of an n-th order MLDE is structured as:

$$\mathcal{D}_k^{(n)} f(\tau) + \phi_2(\tau)\mathcal{D}_k^{(n-2)} f(\tau) + \phi_4(\tau)\mathcal{D}_k^{(n-4)} f(\tau) + \cdots + \phi_n(\tau) f(\tau) = 0 \tag{7.2}$$

where $\mathcal{D}_k^{(m)} = \mathcal{D}_{k+2m-2} \circ \cdots \circ \mathcal{D}_{k+2} \circ \mathcal{D}_k$ represents the iterated modular derivative, and each coefficient $\phi_{2i}(\tau)$ is a true, holomorphic modular form of weight $2i$ on $\mathrm{SL}(2, \mathbb{Z})$.

$$\mathcal{D}_k^{(n)} f(\tau) + \phi_2(\tau)\mathcal{D}_k^{(n-2)} f(\tau) + \phi_4(\tau)\mathcal{D}_k^{(n-4)} f(\tau) + \cdots + \phi_n(\tau) f(\tau) = 0 \tag{7.3}$$

where $\mathcal{D}_k^{(m)} = \mathcal{D}_{k+2m-2} \circ \cdots \circ \mathcal{D}_{k+2} \circ \mathcal{D}_k$ represents the iterated modular derivative, and each coefficient $\phi_{2i}(\tau)$ is a true, holomorphic modular form of weight $2i$ on $\mathrm{SL}(2, \mathbb{Z})$.

7.1.1 The Monic Constraint and Base Forms

Because the vector spaces of holomorphic modular forms of low weights (M_2, M_4, M_6) are explicitly determined by our Eisenstein bases E_4 and E_6 from Chapter 3, the coefficients ϕ_{2i} can be parameterized directly:

- Since $M_2(\mathrm{SL}(2, \mathbb{Z})) = \{0\}$, there is no $\mathcal{D}_k^{(n-1)}$ term in the monic setup.
- The coefficient $\phi_4(\tau)$ must be a multiple of $E_4(\tau)$, meaning $\phi_4 = \alpha E_4$.
- The coefficient $\phi_6(\tau)$ must be a multiple of $E_6(\tau)$, meaning $\phi_6 = \beta E_6$.

In a standard ordinary differential equation, solving for unknown coefficient functions is an **infinite-dimensional problem** because the functional search space spans all arbitrary holomorphic functions, which possess an infinite number of independent degrees of freedom. However, enforcing the constraint that these coefficients must transform as true modular forms completely drops this complexity. Because the coefficients are forced to reside strictly within the low-weight spaces M_2, M_4, and M_6, their possible configurations collapse down into the finite, one-dimensional vector spaces spanned by our explicit bases. This elegant constraint reduces an infinite-dimensional functional search problem down to a simple, finite algebraic calculation of the scalar parameters α and β.

7.2 Detailed Analysis of the Second-Order MLDE

To analyze a concrete mathematical system, we evaluate the general second-order $(n = 2)$ MLDE of weight k. Substituting the basis constraints yields the explicit equation:

$$\mathcal{D}_{k+2}\left(\mathcal{D}_k f(\tau)\right) + \mu E_4(\tau) f(\tau) = 0 \tag{7.4}$$

where $\mu \in \mathbb{C}$ is a free parameter. Expanding this equation in terms of the standard differentiation operator $\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ reveals the raw differential structure:

$$\left[\theta^2 - \frac{k+1}{6}E_2(\tau)\theta + \frac{k(k+2)}{144}E_2^2(\tau) - \frac{k}{12}\theta E_2(\tau) + \mu E_4(\tau)\right] f(\tau) = 0 \tag{7.5}$$

Substituting Ramanujan's differential identity for θE_2 from Chapter 6 ($\theta E_2 = \frac{E_2^2 - E_4}{12}$) into equation (7.3) allows us to simplify and collect terms:

$$\left[\theta^2 - \frac{k+1}{6}E_2(\tau)\theta + \frac{k(k+4)}{144}E_2^2(\tau) + \left(\mu + \frac{k}{144}\right)E_4(\tau)\right] f(\tau) = 0 \tag{7.6}$$

7.3 Solving via the Frobenius Method (q-Expansion)

Because $E_2(\tau) = 1 - 24q - \dots$ and $E_4(\tau) = 1 + 240q + \dots$, we can solve equation (7.4) at the cusp boundary ($\tau \to \infty$, where $q \to 0$) by inserting the Frobenius ansatz for the partition character:

$$f(\tau) = q^h \sum_{m=0}^{\infty} c_m q^m, \quad c_0 \neq 0 \tag{7.7}$$

where h represents the conformal weight of the physical state. Substituting this series back into the full differential equation and gathering the leading indices (q^h) yields the quadratic **Indicial Equation**:

$$h^2 - \frac{k+1}{6}h + \left[\frac{k(k+4)}{144} + \mu + \frac{k}{144}\right] = 0 \tag{7.8}$$

The two roots of this equation, h_1 and h_2, define the baseline scaling exponents for the characters.

7.3.1 Explicit Derivation of the Coefficients and Recursion Relations

To explicitly track how the higher-order terms c_m behave, we substitute the known Fourier q-expansions of our quasi-modular and modular bases into the full differential equation:

$$E_2(\tau) = 1 - 24q - 288q^2 - 864q^3 - \dots = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n \tag{7.9}$$

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \dots = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \tag{7.10}$$

Plugging these expansions alongside our Frobenius ansatz $f(\tau) = \sum_{m=0}^{\infty} c_m q^{m+h}$ into equation (7.4) yields a consolidated series format:

$$\sum_{m=0}^{\infty}(m+h)^2 c_m q^{m+h} - \frac{k+1}{6}\left(1-24q-\dots\right)\sum_{m=0}^{\infty}(m+h)c_m q^{m+h} + \frac{k(k+4)}{144}\left(1-48q-\dots\right)\sum_{m=0}^{\infty} c_m q^{m+h} + \left(\mu+\frac{k}{144}\right)\left(1+240q+\dots\right)\sum_{m=0}^{\infty} c_m q^{m+h} = 0 \tag{7.11}$$

By extracting the lowest power of q ($m = 0$, corresponding to the q^h boundary), we recover the Indicial Equation already detailed in equation (7.6). To solve for all subsequent higher terms ($m \geq 1$), we isolate the matching coefficients for the general power q^{m+h}, which forces the series to balance to zero:

$$\left[(m+h)^2 - \frac{k+1}{6}(m+h) + \frac{k(k+4)}{144} + \mu + \frac{k}{144}\right] c_m = \sum_{j=1}^{m} R_j(m,h,k,\mu)\, c_{m-j} \tag{7.12}$$

where R_j represents the deterministic recurrence kernel tracking the contributions from lower-order states.

7.3.2 The First Two Higher-Order Recursion Terms

By shifting indices and collecting the linear components explicitly, we compute the exact formulas for the first two higher-order terms:

1. **The c_1 Term ($m = 1$ Contribution):**

$$c_1 = \frac{\left[-4(k+1)(h) - \frac{k(k+4)}{3} - 240\left(\mu + \frac{k}{144}\right)\right] c_0}{(1+h)^2 - \frac{k+1}{6}(1+h) + \frac{k(k+4)}{144} + \mu + \frac{k}{144}} \tag{7.13}$$

2. **The c_2 Term ($m = 2$ Contribution):**

$$c_2 = \frac{\left[-4(k+1)(1+h) - \frac{k(k+4)}{3} - 240\Lambda\right] c_1 + \left[-48(k+1)h + \frac{2k(k+4)}{3} - 2160\Lambda\right] c_0}{(2+h)^2 - \frac{k+1}{6}(2+h) + \frac{k(k+4)}{144} + \mu + \frac{k}{144}} \tag{7.14}$$

where $\Lambda = \mu + \frac{k}{144}$.

7.3.3 Physical Admissibility Requirements

In structural applications, lattice networks, or Conformal Field Theory characters, the expanded coefficients c_m serve a rigid physical function: they count the degeneracy of physical states at a given energy level. Because a system cannot possess a fractional or negative number of valid configurations, we enforce the following admissibility constraints:

$$c_m \in \mathbb{Z}_{\geq 0} \quad \forall\, m \geq 1 \tag{7.15}$$

When these explicitly calculated rational fractions for c_1 and c_2 are matched against the non-negative integer requirement, almost all values for the free parameters μ and k are blocked. This proves mathematically why only a highly restricted, discrete set of structures can exist in stable symmetric fields, validating the classification power of MLDE models.

7.3.4 Mapping Recursion Parameters against Valence Formula Bounds

To establish structural consistency across the report, the rational coefficients c_1 and c_2 derived via equations (7.11) and (7.12) must be reconciled with the geometric constraints proved in Chapter 5.

Let the solution $f(\tau) = q^h(c_0 + c_1 q + c_2 q^2 + \dots)$ be a non-zero holomorphic modular form of weight k generated by our second-order MLDE. By definition, as $\tau \to i\infty$ ($q \to 0$), the function vanishes to the leading power h, meaning the order of vanishing at the cusp is given precisely by the indicial root:

$$\mathrm{ord}_\infty(f) = h \tag{7.16}$$

Recalling the Valence Formula ($k/12$ theorem) from equation (5.1), we map this cusp order directly into our boundary inequality constraint:

$$h + \frac{1}{2}\mathrm{ord}_i(f) + \frac{1}{3}\mathrm{ord}_\rho(f) + \sum_{z \in \mathcal{F}, z \neq i, \rho} \mathrm{ord}_z(f) = \frac{k}{12} \tag{7.17}$$

Because $f(\tau)$ is holomorphic across the closure of the fundamental domain, the multiplicities of its internal zeros must strictly be non-negative integers ($\mathrm{ord}_z(f) \geq 0$). Dropping the positive boundary evaluations at i, ρ, and internal points isolates a strict, upper geometric ceiling on our leading Frobenius exponent:

$$h \leq \frac{k}{12} \tag{7.18}$$

7.3.5 Geometric Elimination of Coefficient Divergences

This geometric bound creates a major stabilization mechanism for our recursion algebra:

- **Preventing Pole Singularities:** Looking closely at the denominators of our explicit formulas for c_1 and c_2, the expressions contain polynomials that can drop to zero for certain values of h, causing the recursion to diverge to infinity. Imposing the valence constraint $h \leq k/12$ mathematically ensures that the denominators never cross these singularity zones for valid low-weight systems.
- **Bounding the Character Polynomials:** Because h is strictly bounded by the weight fraction, the numerators of equations (7.11) and (7.12) can only take on a highly finite set of values.

When the valence requirement $h \leq k/12$ is cross-referenced with our physical requirement that $c_1, c_2 \in \mathbb{Z}_{\geq 0}$, the continuous parameter space for the free MLDE parameter μ collapses entirely. For a given weight k, only discrete, solitary values of μ yield admissible integer sequences. This final mathematical mapping provides the exact classification mechanism needed to identify real, stable physical state characters within conformal field theories and metamaterial designs.

Bibliography

Graduate Texts in Mathematics, Vol. 228, Springer-Verlag, New York, 2005.

[1] J.-P. Serre, *A Course in Arithmetic* Graduate Texts in Mathematics, Vol. 7, Springer-Verlag, New York, 1973.

[2] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics, Vol. 41, Springer-Verlag, New York, 1990.

[3] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 1997.

Weisstein, Eric W., "Modular Group Fundamental Domain," *From MathWorld–A Wolfram Web Resource*, 2026.

[4] G. W. Moore and N. Seiberg, "Classical and Quantum Conformal Field Theory," *Communications in Mathematical Physics*, Vol. 123, No. 2, pp. 177–254, 1989.

[5] Y. Arike, M. Kaneko, K. Nagatomo, and Y. Sakai, "Affine Vertex Operator Algebras and Modular Linear Differential Equations," *Letters in Mathematical Physics*, Vol. 106, No. 5, pp. 693–718, 2016.

[6] A. Ramesh Chandra and S. Mukhi, "Towards a Classification of Two-Character Rational Conformal Field Theories," *Journal of High Energy Physics*, Vol. 2019, No. 4, pp. 1–35, 2019.

[7] J. Mohanty, "Modular Linear Differential Equations to Classify Rational Conformal Field Theories," *Indian Institute of Technology - Kanpur, Final Research Repository*, pp. 1–12, 2023.

[8] L. Schneps and B. Verbeek, "Non-holomorphic Modular Forms from Zeta Generators and Iterated Eisenstein Integrals," *Utrecht University & ETH Zurich Research Collection*, pp. 1–48, 2024.