Aspects of *Unitarity* in Black Hole physics

A Study using the Wheeler-deWitt Formalism

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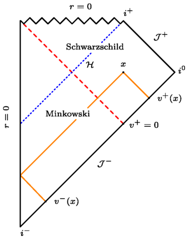
Motivation and Background

- Black holes even though classically are the simplest objects to describe, become the most complex paradoxical systems when quantum processes are considered in their vicinity. This gives rise to the famous *"Hawking Radiation"*. The essential criterion which must be satisfied for Hawking radiation to occur are (see [1]) :-

  1. Presence of a null hypersurface

  2. Non-uniqueness of quantum ground state in the presence of a non-trivial background metric.



Research Question

- The research question we have set out to answer is :-
"Is unitarity preserved in the quantum evolution of a state in the presence of a non-trivial background metric?"
Following [5], we define unitarity in any scattering theory to be a principle of a quantum system which has two completely different connotations :-
1. Conservation of Probability.
2. Evolution of a pure state to a pure state.

The main subject of this work is to study the net outgoing radiation from an infinitesimally thin shell which is collapsing and would ultimately result in the formation of a black hole.

Model and the Action

Semi Classical

- The model we will be studying comprises an **infinitesimally thin collapsing shell**, with **background metric** $g_{\mu\nu}$, which will be **neutral** in one case and **charged** in the other, and **a massless scalar field** Φ whose dynamics we are interested in. The massless scalar field is assumed **to couple to the gravitational field** (originating from the presence of a non-trivial background metric), but **not directly to the shell**. We also have **an observer at** $\mathcal{J}^+$ who is just there to register the outgoing flux with a detector and has weak or **no interaction with "shell-metric-scalar" system**. Also, the observer is assumed **not to significantly** affect **the evolution of the system and similarly for the system vis-a-vis the observer.**

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- The action for the whole system is then given by [3] :-

$$S_{tot} = \int d^4x\sqrt{-g}\left[-\frac{\mathcal{R}}{16\pi} + \frac{1}{2}(\partial_\mu \Phi)^2\right] - \sigma \int d^3\xi\sqrt{-\gamma} + S_{obs} \quad (1)$$

The Wheeler-deWitt Equation

Functional Schrödinger Mechanism

- The Wheeler-deWitt equation for the closed universe to which the above system belongs to is given by [2] :-

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$$H_{obs}\Psi^k_{obs} = i\frac{\partial \Psi^k_{obs}}{\partial t_{obs}} \tag{3}$$

- Then $eq^n(2)$ reduces to :-

$$H_{sys}\Psi^k_{sys} = i\frac{\partial \Psi^k_{sys}}{\partial t_{obs}} \tag{4}$$

Exterior-Schwarzschild

- The metric is divided into three parts :-

$$ds^2_{out} = -\left(1 - \frac{R_s}{R}\right) dt^2 + \left(1 - \frac{R_s}{R}\right)^{-1} dr^2 + r^2 d\Omega^2_2 \; (for \;\; r > R(t)) \quad (5)$$

$$ds^2_{in} = -dT^2 + dr^2 + r^2 d\Omega^2_2 \; (for \;\; r < R(t)) \quad (6)$$

$$ds^2_{on-shell} = -d\tau^2 + r^2 d\Omega^2_2 \; (for \;\; r = R(t)) \quad (as \;\; r \;\; is \;\; constant) \quad (7)$$

- The action for the shell is :-

$$S_{shell} = -4\pi\sigma \int dt \; R^2 \left[\sqrt{B - \frac{R_t^2}{B}} - 2\pi\sigma R \sqrt{B - \frac{1-B}{B} R_t^2} \right] \quad (8)$$

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- The Hamiltonian and The Conjugate Momenta are :-

$$\Pi_{shell} = \frac{\partial \mathcal{L}_{shell}}{\partial R_t} = \frac{4\pi\sigma R^2 R_t}{\sqrt{B}} \left[\frac{1}{\sqrt{B^2 - R_t^2}} - \frac{2\pi\sigma R(1-B)}{\sqrt{B^2 - (1-B)R_t^2}} \right] \tag{9}$$

$$\mathcal{H}_{shell} = 4\pi\sigma B^{3/2} R^2 \left[\frac{1}{\sqrt{B^2 - R_t^2}} - \frac{2\pi\sigma R}{\sqrt{B^2 - (1-B)R_t^2}} \right] \tag{10}$$

- We define the "incipient" limit as the limit where the black hole formation starts, i.e., in the limit $R(t) \to R_s$. Let us present, $\mathcal{H}_{shell}$ and Π_{shell} in this limit.

$$\mathcal{H}_{shell} = \frac{4\pi B^{3/2} \mu R^2}{\sqrt{B^2 - R_t^2}} \tag{11}$$

$$\Pi_{shell} = \frac{4\pi \mu R^2 R_t}{\sqrt{B}\sqrt{B^2 - R_t^2}} \tag{12}$$

where, $\mu := \sigma(1 - 2\pi\sigma R_s)$. So, we have :-

$$\mathcal{H}_{shell} = [(B\Pi_{shell})^2 + B(4\pi\mu R^2)^2]^{1/2} \equiv [p^2 + m^2]^{1/2} \tag{13}$$

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- We also obtain :-

$$R_f = R_s + (R_0 - R_s)e^{-t_f/R_s} \tag{14}$$

- We have the action for Φ as :-

$$S_\Phi = \int d^4x\ r^2 \sin(\theta)[g^{00}(\Phi_0)^2 + g^{11}(\Phi_1)^2] \tag{15}$$

which in the incipient limit becomes :-

$$S_\Phi \to 2\pi \int dt \left[-\frac{1}{B} \int_0^{R_s} dr\ r^2 (\partial_t \Phi)^2 + \int_{R_s}^{\infty} dr\ r^2 \left(1 - \frac{R_s}{r}\right) (\partial_r \Phi)^2 \right] \tag{16}$$

Now, due to the separability property of the Φ equations of motion, we see that the following mode expansion of $\Phi(r,t)$ generally holds :-

$$\Phi(r,t) = \sum_k a_k(t) f_k(r) \tag{17}$$

- Now let us setup S_Φ in terms of the modes a_k, as $R \to R_s$:-

$$S_\Phi = \int dt \sum_{k,k'} \left[-\frac{1}{2B} \frac{da_k}{dt} A_{kk'} \frac{da_{k'}}{dt} + \frac{1}{2} a_k B_{kk'} a_{k'} \right] \tag{18}$$

with the following definitions for $A_{kk'}$ and $B_{kk'}$,

$$A_{kk'} = 4\pi \int_0^{R_s} dr \; r^2 f_k(r) f_{k'}(r) \tag{19}$$

$$B_{kk'} = 4\pi \int_{R_s}^{\infty} dr \; r^2 \left(1 - \frac{R_s}{r}\right) f'_k(r) f'_{k'}(r) \tag{20}$$

- Now, let us define the momentas, π_k conjugate to the modes, a_k by :-

$$\pi_k := \frac{\partial \mathcal{L}_\Phi}{\partial \dot{a_k}} \equiv -i \frac{\partial}{\partial a_k} \tag{21}$$

Then, $\mathcal{L}_\phi$ and $\mathcal{H}_\phi$:-

$$\mathcal{L}_\Phi = -\frac{1}{2B}(\dot{\mathbf{a}}^T \mathbf{A} \dot{\mathbf{a}}) + \frac{1}{2}(\mathbf{a}^T \mathbf{B} \mathbf{a}) \tag{22}$$

$$\mathcal{H}_\Phi = \frac{B}{2}(\mathbf{\Pi}^T \mathbf{A}^{-1} \mathbf{\Pi}) + \frac{1}{2}(\mathbf{a}^T \mathbf{B} \mathbf{a}) \tag{23}$$

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- Thus, we get (for a single eigenbasis, b) :-

$$\left[-\left(1 - \frac{R_s}{R}\right)\frac{1}{2\alpha}\frac{\partial^2}{\partial b^2} + \frac{1}{2}\beta b^2\right]\psi(b,t) = i\frac{\partial \psi(b,t)}{\partial t} \tag{24}$$

- Let us define :-

$$\eta := \int_0^t dt \left(1 - \frac{R_s}{R}\right) \tag{25}$$

$$\Rightarrow \frac{\partial \eta}{\partial t} = B \tag{26}$$

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- With this :-

$$\left[-\frac{1}{2\alpha}\frac{\partial^2}{\partial b^2} + \frac{\beta}{2B}b^2\right]\psi(b,\eta) = i\frac{\partial \psi(b,\eta)}{\partial \eta} \tag{27}$$

Define :-

$$\omega^2(\eta) := \left(\frac{\beta}{\alpha}\right)\frac{1}{B} =: \frac{\omega_0^2}{B} \tag{28}$$

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- Then, $eq^n(24)$ becomes :-

$$\left[-\frac{1}{2\alpha}\frac{\partial^2}{\partial b^2} + \frac{1}{2}\alpha\omega^2(\eta)b^2\right]\psi(b,\eta) = i\frac{\partial \psi(b,\eta)}{\partial \eta} \tag{29}$$

- The solution to $eq^n(29)$ is [4] :-

$$\psi(b,\eta) = e^{i\chi(\eta)} \left[\frac{\alpha}{\pi\zeta^2}\right]^{1/4} \exp\left[i\left(\frac{\zeta_\eta}{\zeta} + \frac{i}{\zeta^2}\right)\frac{\alpha b^2}{2}\right] \qquad (30)$$

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- Where, ζ is the solution of the following differential equation,

$$\zeta_{\eta\eta} + \omega^2(\eta)\zeta = \frac{1}{\zeta^3} \quad (31)$$

with the following initial conditions :-

$$\zeta(0) = \frac{1}{\sqrt{\omega_0}} \quad (32)$$

$$\zeta_\eta(0) = 0 \quad (33)$$

and, $\chi(\eta)$ is given by :-

$$\chi(\eta) := -\frac{1}{2}\int_0^\eta \frac{d\eta'}{\zeta^2(\eta')} \quad (34)$$

- Now, expanding $\psi(b,t)$ in SHO bases we have :-

$$\psi(b,t) = \sum_n c_n(t)\phi_n(b) \tag{35}$$

Calculating the c_n's we obtain :-

$$c_n = \begin{cases} \frac{(-1)^{n/2}e^{i\chi}}{(\Omega_f\zeta^2)^{1/4}}\sqrt{\frac{2}{P}}\left(1-\frac{2}{P}\right)^{n/2}\frac{(n-1)!!}{\sqrt{n!}}, & for \quad even \;\; n \\ 0, & for \quad odd \;\; n \end{cases} \tag{36}$$

where, $P := 1 - \frac{i}{\Omega_f}\left(\frac{\zeta_\eta}{\zeta} + \frac{i}{\zeta^2}\right)$.

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- The density matrices are :-

$$\hat{\rho}_i = \sum_{m,n} I_{mn}|\psi_m\rangle\langle\psi_n| \;,\; \hat{\rho}_f = \sum_{m,n} c_{mn}|\phi_m\rangle\langle\phi_n| \tag{37}$$

$$\Rightarrow Tr(\hat{\rho}_i) = 1 \tag{38}$$

$$\Rightarrow Tr(\hat{\rho}_f) = \frac{2}{\sqrt{\Omega_f\zeta^2|P|}}\frac{1}{\sqrt{1-\left|1-\frac{2}{P}\right|^2}} = 1 \tag{39}$$

- In the Schrödinger picture :-

$$J^0 = |\psi|^2 \ , \ \vec{J} = \frac{1}{2\alpha i}[\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*] = \vec{0} \tag{40}$$

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$$\nabla_\mu J^\mu = \frac{\partial|\psi|^2}{\partial t} = \frac{\partial|\psi|^2}{\partial \eta}\frac{\partial \eta}{\partial t} = B\frac{\partial|\psi|^2}{\partial \eta} \quad (from\ eq^n(3.52))$$

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$$For\ R \to R_s, \ \nabla_\mu J^\mu = 0 \quad (as, \ B \to 0) \tag{41}$$

- Let $t_{obs} = \tau$, then we have :-

$$\nabla_\mu J^\mu = \frac{\partial |\psi|^2}{\partial \tau} = \frac{\partial |\psi|^2}{\partial \eta} \frac{\partial \eta}{\partial \tau} \tag{42}$$

For $R \to R_s$, $\frac{\partial \eta}{\partial \tau} = R_\tau|_{R \to R_s}$ (as, $B \to 0$). Thus,

$$\nabla_\mu J^\mu = \frac{\partial |\psi|^2}{\partial \eta} \ R_\tau|_{R \to R_s} = 0 \tag{43}$$

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- Using above models to try and resolve questions regarding first quantization i.e. how to move from a symplectic manifold to a Hilbert space representation? Also, does ordering implies unitarity in a quantum system?

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