Notion of *Temperature* in General Relativity - A Brief Study.

Summer Project Report

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Abstract

In this project we have studied how to define temperature in general relativity by which we mean, how to define temperature of black holes using the concept of surface gravity of black holes and associating it with the Hawking Temperature. Then we went on to study the Raychaudhuri Equation and Zeroth Law of Black Hole Mechanics. Finally, we computed surface gravities for some already known stationary black hole solutions and ultimately moved onto the realm of higher dimensional ($d > 4$) black holes and stationary black ring solutions (some of them) and computed their temperatures too.

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Chapter 1

Notion of Manifold

We begin by presenting in this and few of the subsequent chapters; some mathematical tools of *Differential Geometry* needed for the understanding of the concept of *surface gravity* and related concepts often associated with black hole physics.

1.1 Manifold

Before rigorously defining a manifold, we would like to state that a manifold of dimension n is intuitively an entity which is locally Euclidean or in other words, whose certain class of subsets (locally observing the manifold) are in one-to-one correspondence with open subsets of $\Re^n$.

Definition 1.1.1. A set $\emptyset \neq M$ is said to be a n-dimensional $C^\infty - real\ manifold$ along with a collection of subsets $\{O_\alpha\}_{\alpha\epsilon I}$ $(I - index\ set)$, if it satisfies the following property :-
i) $M = \cup_{\alpha\epsilon I}\ O_\alpha$.
ii) $\forall\ \alpha\epsilon I\ \exists\ a\ bijection\ \psi_\alpha : O_\alpha \to U_\alpha \subseteq_{open} \Re^n$.
iii) Let O_α and O_β be two subsets of M such that $O_\alpha \cap O_\beta \neq \emptyset$. Now, consider the map;
$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(O_\alpha \cap O_\beta) \to \psi_\beta(O_\alpha \cap O_\beta)$ where $\psi_\alpha(O_\alpha \cap O_\beta) \subseteq U_\alpha \subseteq \Re^n$ and $\psi_\beta(O_\alpha \cap O_\beta) \subseteq U_\beta \subseteq \Re^n$. In the above consideration we demand the map defined to be of the class of C^∞ and both the domain and range of the map to be *open* of $\Re^n$.

Remark. The ψ_α are called as coordinate chart or coordinate system of the manifold.
$\{\psi_\alpha\}_{\alpha\epsilon I}$ is an atlas.

Note. Demanding $\forall\ \alpha\epsilon I$, ψ_α to be *homeomorphisms* we can define a *topology* on M.

Remark. From here on the word manifold would mean a $C^\infty -\ real\ manifold$.
Also wherever the word smooth map would be used it would mean the map belongs to class of C^∞ maps.
Further, *Einstein summation convention* is assumed and calculations are done in *natural units*.

1.1.1 Product Manifold

Definition 1.1.2. Let M and $M^{'}$ be two $n, n^{'}$-dimensional manifolds with atlases as $\{\psi_\alpha\}_{\alpha\epsilon I}$ and $\{\psi^{'}_\beta\}_{\beta\epsilon I^{'}}$. Now, $M \otimes M^{'}$ consists of points $(p, p^{'})$ where $p\epsilon M$ and $p^{'}\epsilon M^{'}$. Further, $\psi_\alpha : O_\alpha \to U_\alpha \subseteq_{open} \Re^n$ and $\psi^{'}{}_\beta : O^{'}{}_\beta \to U^{'}{}_\beta \subseteq_{open} \Re^{n^{'}}$. We define $M \otimes M^{'}$ to be the product manifold with atlas $\{\psi_{\alpha\beta}\}_{\alpha\epsilon I,\ \beta\epsilon I^{'}}$ where, $\psi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta}$ and, $O_{\alpha\beta} = O_\alpha \otimes O^{'}{}_\beta$; also, $U_{\alpha\beta} = U_\alpha \otimes U^{'}{}_\beta \subseteq_{open} \Re^{n+n^{'}}$ as:-
$\psi_{\alpha\beta}(p, p^{'}) = (\psi_\alpha(p), \psi^{'}{}_\beta(p^{'}))$.

Remark. $M \otimes M^{'}$ with above defined atlas can be checked to be a manifold indeed.
This particular way of construction of manifolds given a number of manifolds is particularly useful in physics as we will see in the following sections where we would see that event horizons of black ring solutions are of $S^1 \otimes S^2$ topology.

1.1.2 Differentiability on Manifold

Definition 1.1.3. Let M and $M^{'}$ be two manifolds as defined in the preceding section. Let $f : M \to M^{'}$ be a map. It is said to be $differentiable$ iff the map $\psi^{'}{}_\beta \circ f \circ \psi_\alpha^{-1}$ is differentiable.

1.2 Tangent Plane

Before going into the concept of tangent plane, we would like to introduce the concept of tangent vectors by using the notion of differentiability on manifolds we presented in the preceding section. We start by stating a theorem of multivariable calculus.

Theorem 1.2.1. *In $\Re^n$ $\exists$ a one-to-one correspondence between vectors and directional derivatives. In other words, a vector $v = (v^1, v^2, \cdots\cdots, v^n)$ defines a directional derivative operator $v^\mu \frac{\partial}{\partial x^\mu}$ (where x^μ's are the Cartesian coordinates of $\Re^n$) and vice-versa.*

Now, directional derivative operator are linear and obey the *Leibnitz rule* when acting on functions. Motivated by this we define a tangent vector on a manifold.

Definition 1.2.1. Let M be a n-dimensional manifold and $p\epsilon M$. Let $\mathcal{F} = \{f | f : M \to \Re \ is \ smooth\}$. Define a map, $v : \mathcal{F} \to \Re$ such that it satisfied the following properties:-
i) Linearity - $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g) \ \forall \ \alpha, \beta \epsilon \Re; \ f, g \epsilon \mathcal{F}$.
ii) Leibnitz rule - $v(fg) = fv(g) + gv(f) \ \forall \ f, g\epsilon\mathcal{F}$
where, $v(f) = v(f(p))$. Then, v is said to be a tangent vector at the point p.

This definition leads us to the following lemma.

Lemma 1.2.2. *Let M be a n-dimensional manifold and $p\epsilon M$. Let $\mathcal{F} = \{f | f : M \to \Re \ is \ smooth\}$. Let $h\epsilon\mathcal{F}$. Now, let $h(p) = c(const)$. Then, $v(h) = 0$.*

Proof. *Since $h(p) = c$, $h(h(p)) = h(c) = ch = h^2$. So, $v(h^2) = v(ch) = cv(h)$.
Alternatively consider, $v(h^2) = hv(h) + hv(h) = 2cv(h)$. This gives $2cv(h) = cv(h)$.
Hence, we obtain the result, $v(h) = 0$.* **Proved**.

Definition 1.2.2. Let $V_p = \{v | v \ is \ a \ tangent \ vector \ at \ p\}$. Then, V_p is called the tangent plane at p.

Note. V_p is a vector space over the field $\Re$ under :-
i) Addition - $(v_1 + v_2)(f) = v_1(f) + v_2(f) \ \forall \ v_1, v_2\epsilon V_p; \ f\epsilon\mathcal{F}$.
ii) Scalar Multiplication - $(av)(f) = av(f) \ \forall \ v\epsilon V_p; \ a\epsilon\Re; \ f\epsilon\mathcal{F}$.

V_p can be indeed shown to be an inner-product space.

Another useful property of V_p is the content of the following theorem.

Theorem 1.2.3. *$dim(V_p) = n$.*

Proof. *We prove the theorem by constructing n linearly independent tangent vectors that span V_p.
Let $\psi : O \to U$ be a chart with $p\epsilon O$ and let $f\epsilon\mathcal{F}$. Now since both f and ψ^{-1} are smooth so $f \circ \psi^{-1} : U \to \Re$ is smooth. $\forall$ $\mu = 1, 2, \cdots\cdots, n$ define; $X_\mu : \mathcal{F} \to \Re$ as :-
$X_\mu(f) = \left| \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \right|_{\psi(p)}$.
Consider the set $\{X_\mu\}$. Clearly, the set consists of linearly independent tangent vectors as partial derivative in linear and follows Leibnitz rule.*

To prove that this set spans V_p consider Taylor's theorem in $\Re^n$. It states :-

*"$If \ F : \Re^n \to \Re \ is \ smooth, \ then \ for \ each \ a\epsilon\Re^n \ \exists \ smooth \ functions \ H_\mu \ such \ that \ \forall \ x\epsilon\Re^n, \ we \ have; -$
$F(x) = F(a) + (x^\mu - a^\mu)H_\mu(x)$.
$Furthermore,$
$H_\mu(a) = \left| \frac{\partial F}{\partial x^\mu} \right|_{x=a}$."*

Now in the above result letting $F = f \circ \psi^{-1}$ and $a = \psi(p)$, we have $\forall$ $q\epsilon O$:-

$$f(q) = f(p) + (x^\mu \circ \psi(q) - x^\mu \circ \psi(p))H_\mu(\psi(q)) \tag{1.1}$$

*Let $v\epsilon V_p$ act on f. Then from Eq^n 1.1 we get using Leibnitz rule;
$v(f) = v(f(p)) + |(x^\mu \circ \psi(q) - x^\mu \circ \psi(p))|_{q=p} v(H_\mu \circ \psi) + |(H_\mu \circ \psi)|_p v(x^\mu \circ \psi(q) - x^\mu \circ \psi(p))$.
Since, by Lemma 1.2.2, we have $v(f(p)) = 0$, we get :-
$v(f) = (H_\mu \circ \psi(p))v(x^\mu\psi)$.
Now, by using the form of H_μ in Taylor's theorem we see, $H_\mu(\psi(p)) = X_\mu(f)$.
Thus, $\forall$ $f\epsilon\mathcal{F}$ we have :-*

$$v(f) = v^\mu X_\mu(f) \tag{1.2}$$
$$v = v^\mu X_\mu \tag{1.3}$$

where,

$$v^\mu = v(x^\mu \circ \psi) \tag{1.4}$$

So, the set constructed indeed spans V_p. Hence, $dim(V_p) = n$. **Proved**.

The basis constructed above is known as the coordinate basis. Coordinate chart ψ gave rise to basis $\{X_\mu\}$ while choosing a different coordinate chart $\psi^{'}$ would give rise to new basis $\{X^{'}{}_\nu\}$. By using chain rule one obtains :-

$$X_\mu = \frac{\partial x^{'\nu}}{\partial x^\mu} X^{'}{}_\nu \tag{1.5}$$

where, $x^{'\nu}$ denotes the ν^{th} component of the map $\psi^{'} \circ \psi^{-1}$.
Thus, by using $Eq^n s$ 1.3 and 1.5 we see that :-

$$v^{'\nu} = \frac{\partial x^{'\nu}}{\partial x^\mu} v^\mu \tag{1.6}$$

Above is the *vector transformation law.*

1.2.1 Tangent to a Curve on a Manifold

Lets define a curve on a manifold first.

Definition 1.2.3. A smooth curve C on a manifold is a smooth map $C : \mathcal{I} \to M$, where; $\mathcal{I}$ is an open interval of $\Re$.

Now let us introduce the concept of tangent vector to a curve.

Definition 1.2.4. Let $\psi : O \to U$ be a chart and $p \epsilon C$ and O is a *nbd* of p. The tangent vector at p is defined as :-
$T^a : \mathcal{F} \to \Re$ by :-

$$T^a(f) = \frac{d}{d\lambda}(f \circ C) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda} X_\mu(f) \tag{1.7}$$

where λ is a parameter for the curve and $\lambda \epsilon \mathcal{I}$.

So, components of tangent vector to the curve are :-

$$T^\mu = \frac{dx^\mu}{d\lambda} \tag{1.8}$$

1.2.2 Affine Parametrization of a Curve

Definition 1.2.5. A curve C on a manifold M is said to be affinely parametrized by affine paramete λ if the norm of its tangent vector at every point on the curve is independent of the parameter i.e.;

$$T^\mu = \frac{dx^\mu}{d\lambda} = const \tag{1.9}$$

1.2.3 Tangent Vector Field

In all the above discussions we have focused on a fixed point $p \epsilon M$ and defined the tangent plane V_p at that point. Similarly, we can consider another point on the manifold $p \neq q \epsilon M$ and can in a similar fashion define its respective tangent plane V_q. Now, V_p should be identified with V_q in order to find the rate of change of a vector along a curve. But there is no immediate identification of V_p with V_q. In $\Re^n$ this identification is trivial. We will consider this general identification in following sections while discussing the concept of *parallel transport* in which the identification is unique upto a given curve.

Definition 1.2.6. A *tangent vector field* v on a manifold M is an assignment of a tangent vector, $v_{\downarrow p} \epsilon V_p$, at each point $p \epsilon M$.

Despite the fact that V_p is different from V_q $\exists$ a natural notion of v varying smoothly from point to point on M. If $f \epsilon \mathcal{F}$ is smooth on M then $v(f)_{\downarrow p}$ is a number i.e. $v(f)$ is a function on M. The tangent field v is said to be smooth if for each smooth function f on M, the function $v(f)$ is also smooth.

Since, the coordinate basis fields X_μ are smooth, it follows that a vector field v is smooth iff its coordinates basis components, v^μ, are smooth functions.

Commutator of Vector Fields

Let v, w be smooth vector fields on a manifold M and $f \epsilon \mathcal{F}$. Their commutator is defined as :-

$$[v, w](f) = (v \circ w)(f) - (w \circ v)(f) \tag{1.10}$$

Now we present a useful lemma regarding commutator of vector fields.

Lemma 1.2.4. *If v, w are smooth vector fields occurring in coordinate bases then their commutator vanishes.*

Proof. *Given; $v = X_\mu$ and $w = X_\nu$. Now, explicit calculation shows;*

$$\begin{aligned} v(w(f)) &= X_\mu(X_\nu(f)) \\ &= X_\mu \left(\frac{\partial}{\partial x^\nu}(f \circ \psi^{-1}) \right) \\ &= \frac{\partial^2}{\partial x^\mu \partial x^\nu}(f \circ \psi^{-1}) \end{aligned}$$

$$\begin{aligned} w(v(f)) &= X_\nu(X_\mu(f)) \\ &= X_\nu \left(\frac{\partial}{\partial x^\mu}(f \circ \psi^{-1}) \right) \\ &= \frac{\partial^2}{\partial x^\nu \partial x^\mu}(f \circ \psi^{-1}) \end{aligned}$$

This shows from Eq^n 1.9;

$$[v, w](f) = 0 \qquad \textbf{Proved}.$$

1.3 Metric Tensor

1.3.1 Linear Functionals and Dual Spaces

We start this section by introducing the notions of *linear functional* and *Dual Space*.

Definition 1.3.1. Let V be a finite dimensional vector space over the field $\Re$. Let $T : V \to \Re$ be a linear map. Then T is called a linear functional.

Definition 1.3.2. Let $V^* = \{T | T : V \to \Re\}$. Then, this set of all linear functionals on V is termed as the Dual Space of V.

Note. One check that V^* is indeed a vector space. Elements of V^* are called dual vectors.

If $\{v_\nu\}$ is a basis of V then we can define $\{v^{*\mu} \epsilon V^*\}$ by :-

$$v^{*\mu}(v_\nu) = \delta^\mu{}_\nu \tag{1.11}$$

where $\delta^\mu{}_\nu$ is the Kronecker Delta.

Indeed $\{v^{*\mu}\}$ is a basis of V^*, called the dual basis to the basis $\{v_\mu\}$ of V. In particular, $dim(V^*) = dim(V)$. The correspondence $v_\mu \leftrightarrow v^{*\mu}$ gives rise to an *isomorphism* between V and V^*, but this *isomorphism* depends on the choice of basis $\{v_\mu\}$.

We now can apply the above construction starting with the vector space V^*, thereby obtaining the double dual vector space to V, denoted V^{**}. A vector $v^{**} \epsilon V^{**}$, is a linear map from $v^{**} : V^* \to \Re$. However, V^{**} is naturally isomorphic to the original vector space V. To each vector $v \epsilon V$ we can associate the map in V^{**} whose value on the vector $\omega^* \epsilon V^*$ is just $\omega^*(v)$. In this way, we obtain a $1-1$ linear map of V into V^{**} which must be onto since $dimV = dimV^{**}$. Thus, taking the double dual gives nothing new; we can naturally identify V^{**} with the original vector space V. With this we are ready to define *tensors* in the next section.

1.3.2 Tensors

Definition 1.3.3. Let V be a finite dimensional vector space and let V^* denote its dual vector space. A tensor, T, of type (k, l) over V is a multilinear map;
$T : V^* \otimes V^* \otimes \cdots\cdots \otimes V^* \otimes V \otimes V \otimes \cdots\cdots \otimes V \to \Re$
where V^* occurs k times and V occurs l times.

Note. Let $\mathcal{T}(k,l) = \{T | T : V^* \otimes V^* \otimes \cdots\cdots \otimes V^* \otimes V \otimes V \otimes \cdots\cdots \otimes V \to \Re\}$. With usual rules for addition and scalar multiplication for maps one can show $\mathcal{T}(k,l)$ has the structure of a vector space.

Remark. Since there are n^{k+l} independent ways of filling the slots of a tensor of type (k,l) with such basis vectors (where $n = dimV = dimV^*$); $dim(\mathcal{T}(k,l)) = n^{k+l}$.

Operations on Tensors

We describe two useful operations on tensors.

Contraction

Definition 1.3.4. Let $T \epsilon \mathcal{T}(k,l)$. The contraction of T w.r.t i^{th} (dual vector) and j^{th} (vector) slots is a map $C : \mathcal{T}(k,l) \to \mathcal{T}(k-1, l-1)$ defined as :-

$$CT = T(\cdots\cdots, v^{*\sigma}, \cdots\cdots; \cdots\cdots, v_\sigma, \cdots\cdots) \tag{1.12}$$

where $\{v_\sigma\}$ is a basis for V and $\{v^{*\sigma}\}$ is its dual basis; and these are inserted in the j^{th} and the i^{th} slots respectively.

Outer Product

Given a tensor $T \epsilon \Bbbk(k,l)$ of type and another tensor $T' \epsilon \mathcal{T}(k', l')$ we can construct a new tensor of type $(k+k', l+l')$ called the outer product of T and T' and denoted $T \otimes_k T'$ by the following simple rule. Given $(k+k')$ dual vectors $\{v^{*k+k'}\}$ and $(l+l')$ vectors $\{w_{l+l'}\}$, we define $T \otimes_k T'$ acting on these vectors to be the product of $T(v^{*1}, v^{*2}, \cdots\cdots, v^{*k}; w_1, w_2, \cdots\cdots, w_l)$ and $T(v^{*k+1}, v^{*k+2}, \cdots\cdots, v^{*k+k'}; w_{l+1}, w_{l+2}, \cdots\cdots, w_{l+l'})$.

Thus, one way of constructing tensors is to take outer products of vectors and dual vectors. A tensor which can be expressed as such an outer product is called simple. If $\{v_\nu\}$ is a basis of V and $\{v^{*\mu}\}$ is its dual basis, then n^{k+l} simple tensors $\{v_{\mu_1} \otimes_k v_{\mu_2} \otimes_k \cdots\cdots \otimes_k v_{\mu_k} \otimes_k v^{*\nu_1} \otimes_k v^{*\nu_2} \otimes_k \cdots\cdots v^{*\nu_l}\}$ yields a basis of $\mathcal{T}(k,l)$. Thus, every tensor $T \epsilon \Bbbk(k,l)$ can be expressed as a sum of simple tensors in this collection as :-

$$T = T^{\mu_1\mu_2\cdots\cdots\mu_k}{}_{\nu_1\nu_2\cdots\cdots\nu_l} v_{\mu_1} \otimes_k v_{\mu_2} \otimes_k \cdots\cdots \otimes_k v_{\mu_k} \otimes_k v^{*\nu_1} \otimes_k v^{*\nu_2} \otimes_k \cdots\cdots v^{*\nu_l} \tag{1.13}$$

The basis expansion coefficients $T^{\mu_1\mu_2\cdots\cdots\mu_k}{}_{\nu_1\nu_2\cdots\cdots\nu_l}$ are called the components of the tensor T. In terms of component;

$$Contraction \ - \ (CT)^{\mu_1\mu_2\cdots\cdots\mu_{k-1}}{}_{\nu_1\nu_2\cdots\cdots\nu_{l-1}} = T^{\mu_1\mu_2\cdots\sigma\cdots\mu_{k-1}}{}_{\nu_1\nu_2\cdots\sigma\cdots\nu_{l-1}} \tag{1.14}$$

$$Outer\ Product \ - \ S = T \otimes_k T' = S^{\mu_1\mu_2\cdots\cdots\mu_{k+k'}}{}_{\nu_1\nu_2\cdots\cdots\nu_{l+l'}} = T^{\mu_1\mu_2\cdots\cdots\mu_k}{}_{\nu_1\nu_2\cdots\cdots\nu_l} T'^{\mu_{k+1}\mu_{k+2}\cdots\cdots\mu_{k+k'}}{}_{\nu_{l+1}\nu_{l+2}\cdots\cdots\nu_{l+l'}} \tag{1.15}$$

The above discussion applies to an arbitrary finite dimensional vector space V. Let us now consider the case where $V = V_p$. In this case, ${V^*}_p$ is commonly called the cotangent space and vectors in ${V^*}_p$ are called cotangent vectors. We also commonly refer to vectors in V_p as contravariant vectors and vectors in ${V^*}_p$ as covariant vectors as they transform like the basis vectors (hence the name covariant) while the former like the dual basis vectors. Basis of V_p is $\{\frac{\partial}{\partial x^\nu}\}$ and basis of ${V^*}_p$ is denoted by $\{dx^\mu\}$ (as these are just differentials transforming like contravariant vectors). Furthermore;

$$dx^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu \tag{1.16}$$

Now,

$$T'^{\mu'_1\mu'_2\cdots\cdots\mu'_k}{}_{\nu'_1\nu'_2\cdots\cdots\nu'_l} = \frac{\partial x'^{\mu'_1}}{\partial x^{\mu_1}} \frac{\partial x'^{\mu'_2}}{\partial x^{\mu_2}} \cdots\cdots \frac{\partial x'^{\mu'_k}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{x'^{\nu'_1}} \frac{\partial x^{\nu_2}}{x'^{\nu'_2}} \cdots\cdots \frac{\partial x^{\nu_l}}{x'^{\nu'_l}} T^{\mu_1\mu_2\cdots\cdots\mu_k}{}_{\nu_1\nu_2\cdots\cdots\nu_l} \tag{1.17}$$

Eq^n 1.17 is the *tensor transformation law.*

Definition 1.3.5. An assignment of a tensor over V_p for each point p in the manifold M is called a tensor field.

A tensor $T \epsilon \mathcal{T}(k,l)$ is said to be smooth if for all smooth covariant vectors fields $\omega^1, \omega^2, \cdots\cdots, \omega^k$ and smooth contravariant vector fields $v_1, v_2, \cdots\cdots, v_l$; $T(\omega^1, \omega^2, \cdots\cdots, \omega^k; v_1, v_2, \cdots\cdots, v_l)$ is a smooth function. is a smooth function.

1.3.3 Metric

Intuitively, a metric is supposed to tell us the "infinitesimal squared distance" associated with an *infinitesimal displacement*. The intuitive notion of an *infinitesimal displacement* is precisely captured by the concept of a tangent vector. Thus, since *infinitesimal squared distance* should be quadratic in the displacement, a metric, g, should be a linear map from $g : V_p \otimes V_p \to \Re$ i.e. a tensor of type $(0,2)$. Further, the metric should satisfy the following properties :-
i) Symmetry - $\forall\ v_1, v_2 \epsilon V_p$ we have $g(v_1, v_2) = g(v_2, v_1)$.
ii) Non-degenrate - If $\forall\ v \epsilon V_p$ and given $v_1 \epsilon V_p$ $g(v, v_1) = 0$ then $v_1 = 0$.

Thus, a metric, g, on a manifold M is a symmetric, nondegenerate tensor field of type $(0,2)$.

In coordinate basis, g can be expanded in terms of its components $g_{\mu\nu}$ as :-

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{1.18}$$

This is often written as :-

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.19}$$

Given a metric g, we always can find an *orthonormal basis* $\{v_\mu\}$ of V_p (by *Grahm-Schmidt* orthogonalization), i.e., a basis such that; $g(v_\mu, v_\nu) = 0$ if $\mu \neq \nu$ and $g(v_\mu, v_\mu) = \pm 1$.

Now, $g^{\mu\nu}$ is the matrix inverse of $g_{\mu\nu}$.

With this we can define the inner product between two vectors u, v as :-

$$u^\mu v_\mu = g_{\mu\nu} u^\mu v^\nu \tag{1.20}$$

Signature of Metric

If $g(v_\mu, v_\mu) = 1$ then metric is said to be $positive - definite$ and is called *Riemannian* while if $g(v_\mu, v_\mu) = -1$ the metric is a spacetime metric and is called *Lorentzian*.

1.3.4 Symmetric and Antisymmetric Tensors

For a tensor $T_{a_1, a_2, \cdots\cdots, a_l}$ of type $(0,l)$ we have :-

$$T_{(a_1, a_2, \cdots\cdots, a_l)} = \frac{1}{l!} \sum_\pi T_{a_{\pi(1)}, a_{\pi(2)}, \cdots, \cdots, a_{\pi(l)}} \tag{1.21}$$

$$T_{[a_1, a_2, \cdots\cdots, a_l]} = \frac{1}{l!} \sum_\pi \delta_\pi T_{a_{\pi(1)}, a_{\pi(2)}, \cdots, \cdots, a_{\pi(l)}} \tag{1.22}$$

where the sum is taken over all permutations, π, of $1, 2, \cdots, l$ and $\delta_\pi = +1$ for even permutations and $\delta_\pi = -1$ for odd permutations.

Chapter 2

Curvature

Intuitively, we seek to understand the notion of curvature. They are of types; extrinsic curvature and intrinsic curvature. The former determines how curved a manifold is when it is embedded in a higher-dimensional manifold and w.r.t it. But the notion of intrinsic curvature determines how curved a manifold is intrinsically without reference to anything external. This is extremely useful as one can observe in the def^n of manifold that a manifold to start with with all generality is never considered to be the subset of any other set which is aptly justified for applications to our universe as it is not embedded in any external higher dimensional universe. So, from here on by curvature we mean only intrinsic curvature.

Now, one interpretation of curvature is linked to the parallel transport of a vector along a closed curve on a surface. Since, intuitively we know that if the vector returns to its initial direction then the surface is not curved.

Another interpretation is linked to concept of *geodesics* to be rigorously defined later in this chapter. A geodesic is the straightest possible curve on a manifold. If a family of geodesics initially parallel remain parallel throughout the submanifold spanned by them, then the curvature vanishes, else not.

We will build upon these intuitions in this section. Now, we seek to define a derivative operator ∇ on a manifold M.

2.1 Derivative Operator

Definition 2.1.1. A derivative operator ∇ also known as *covariant derivative* on a manifold M is a map $\nabla : \mathcal{T}(k,l) \to \mathcal{T}(k,l+1)$ satisfying the following properties :-
i) Linearity - $\nabla_a(\alpha T + \beta T^{'}) = \alpha(\nabla_a T) + \beta(\nabla_a T^{'})\ \forall\ T, T^{'} \epsilon \mathcal{T}(k,l);\ \alpha, \beta \epsilon \Re$.
ii) Leibnitz rule - $\nabla_a(TT^{'}) = T\nabla_a T^{'} + T^{'}\nabla_a T\ \forall\ T, T^{'} \epsilon \mathcal{T}(k,l)$.
iii) Commutativity with Contraction - $\nabla_a(C,T) = C(\nabla_a T)\ \forall\ T \epsilon \mathcal{T}(k,l)$ and C is the contraction map.
iv) Consistency with notion of tangent vectors as directional derivatives of scalar fields - $\forall$ smooth functions $f \epsilon \mathcal{F}$ and $\forall\ t \epsilon V_p$ given $p \epsilon M$ we have :- $t(f) = t^a \nabla_a f$.
v) Torsion free - $\nabla_a \nabla_b f = \nabla_b \nabla_a f$.

Note. Due to $cond^n$ *iv)* $\nabla_a f = \tilde{\nabla}_a(f)$ at given $p \epsilon M$, where $\tilde{\nabla}_a$ is a different derivative operator than ∇_a.

Remark. Existence of derivative operator - Take a coordinated basis on M and then one can show that $\partial_a = \frac{\partial}{\partial x^a}$ is indeed a derivative operator satisfying above conditions.

2.1.1 Commutators Revisited

Consider two smooth vector fields v, w and let $f \epsilon \mathcal{F}$. Now,

$$\begin{aligned}
[v,w](f) &= v(w(f)) - w(v(f)) \\
&= v^a \nabla_a (w^b \nabla_b(f)) - w^a \nabla_a (v^b \nabla_b(f)) \quad (by\ cond^n\ iv)) \\
&= \nabla_b(f) v^a \nabla_a(w^b) + v^a w^b \nabla_a(\nabla_b f) - \nabla_b(f) w^a \nabla_a(v^b) - w^a v^b \nabla_a(\nabla_b f) \\
&= \nabla_b(f) v^a \nabla_a(w^b) - \nabla_b(f) w^a \nabla_a(v^b) \quad (by\ cond^n\ v)) \\
&= (v^a \nabla_a w^b - w^a \nabla_a v^b) \nabla_b(f)
\end{aligned}$$

So,

$$[v, w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b \tag{2.1}$$

2.1.2 Christoffel Symbol

Consider,

$(\tilde{\nabla}_a - \nabla_a)(f w_b) = f(\tilde{\nabla}_a w_b - \nabla_a w_b)$; by note to *cond*^{n} iv).

So, given $p \epsilon M$ we can observe that the map $(\tilde{\nabla}_a - \nabla_a)$ takes dual vectors at p to tensors of type $(0, 2)$ at p and its action is a type $(1, 2)$ tensor. So, $\tilde{\nabla}_a w_b - \nabla_a w_b = C^c_{ab} w_c$. Hence,

$$\nabla_a w_b = \tilde{\nabla}_a w_b - C^c_{ab} w_c \tag{2.2}$$

Now let for some $f \epsilon \mathcal{F}$,

$$\begin{aligned} w_b = \nabla_b(f) &= \tilde{\nabla}_b(f) \\ \Rightarrow \nabla_a \nabla_b(f) &= \tilde{\nabla}_a \tilde{\nabla}_b(f) - C^c_{ab} \nabla_c(f) \\ But,\ \nabla_b \nabla_a(f) &= \tilde{\nabla}_b \tilde{\nabla}_a(f) - C^c_{ba} \nabla_c(f) \end{aligned}$$

So, this gives :-

$$C^c_{ab} = C^c_{ba} \tag{2.3}$$

Furthermore, for dual vector field w_b and vector field t^b we have :-

$$(\tilde{\nabla}_a - \nabla_a)(t^b w_b) = 0$$

This gives;

$$\begin{aligned} &\Rightarrow t^b \tilde{\nabla}_a w_b + w_b \tilde{\nabla}_a t^b - t^b \nabla_a w_b - w_b \nabla_a t^b = 0 \\ &\Rightarrow t^b (\tilde{\nabla}_a - \nabla_a) w_b + w_b (\tilde{\nabla}_a - \nabla_a) t^b = 0 \\ &\Rightarrow t^b C^c_{ab} w_c + w_b (\tilde{\nabla}_a - \nabla_a) t^b = 0 \\ &\Rightarrow t^c C^b_{ac} w_b + w_b (\tilde{\nabla}_a - \nabla_a) t^b = 0 \\ &\Rightarrow w_b [(\tilde{\nabla}_a - \nabla_a) t^b + C^b_{ac} t^c] = 0 \end{aligned}$$

So,

$$\nabla_a t^b = \tilde{\nabla}_a t^b + C^b_{ac} t^c \tag{2.4}$$

Similarly, it can be shown that by induction :-

$$\nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d_{ab} g_{dc} - C^d_{ac} g_{bd} \tag{2.5}$$

Now, if $\tilde{\nabla}_a = \partial_a$ then, $C^c_{ab} = \Gamma^c_{ab}$ where, Γ^c_{ab} is called the *Christoffel Symbol.* Then,

$$\nabla_a t^b = \tilde{\nabla}_a t^b + \Gamma^b_{ac} t^c \tag{2.6}$$

2.2 Parallel Transport

Let γ be a curve parametrized by $\lambda \epsilon \Re$ on a manifold M with tangent vector t and let $p \epsilon M$. A vector w is said to be parallely transported along γ if :-

$$t^a \nabla_a w^b = 0 \tag{2.7}$$

$$\Rightarrow t^a \partial_a w^b + t^a \Gamma^b_{ac} w^c = 0 \tag{2.8}$$

$$\Rightarrow \frac{dw^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} w^\rho = 0 \tag{2.9}$$

The meaning of the *Parallel Transport equation* will be clear in the following examples.

Example 2.2.1. Consider $\Re^2$ in which consider the flat metric in polar form given as :-

$$ds^2 = dr^2 + r^2 d\theta^2$$

Consider the following curve on $\Re^2$:-

$$\begin{aligned} C(\lambda) :- \ r(\lambda) &= 1 \\ \theta(\lambda) &= \lambda \end{aligned}$$

where $\lambda \epsilon \Re$. $C(\lambda)$ describes the unit circle in $\Re^2$. Now, let us compute the non-vanishing components of the *Affine Connection.*

$$\begin{aligned} \Gamma^r_{\theta\theta} &= -r \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \frac{1}{r} \end{aligned}$$

Now let us compute the tangent vector fields to the curve $C(\lambda)$.

$$\begin{aligned} t^r &= \frac{dr}{d\lambda} = 0 \\ t^\theta &= \frac{d\theta}{d\lambda} = 1 \\ So,\ t^\mu t_\mu &= 1 \end{aligned}$$

So, given curve $C(\lambda)$ is affinely parametrized with affine parameter λ. Now, we would like to parallely transport a vector u along curve $C(\lambda)$. So, let us set up the parallel transport equations for this.

$$\begin{aligned} &\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} t^\nu u^\rho = 0 \\ \Rightarrow &\frac{du^r}{d\lambda} + \Gamma^r_{\theta\theta} t^\theta u^\theta = 0 \\ \Rightarrow &\frac{du^r}{d\lambda} - u^\theta = 0 \\ Similarly, &\frac{du^\theta}{d\lambda} + \Gamma^\theta_{\theta r} t^\theta u^r + \Gamma^\theta_{r\theta} t^r u^\theta = 0 \\ \Rightarrow &\frac{du^\theta}{d\lambda} + u^r = 0 \end{aligned}$$

Now consider,

$$\frac{d^2u^r}{d\lambda^2} - \frac{du^\theta}{d\lambda} = 0 = \frac{d^2u^r}{d\lambda^2} + u^r$$

Solution of the above equation is :-

$$u^r(\lambda) = A\cos\lambda + B\sin\lambda$$

Hence,

$$u^\theta(\lambda) = B\cos\lambda - A\sin\lambda$$

where $A, B \epsilon \Re$.

Now consider the given initial conditions for u :-

$$\begin{aligned} u^r(0) &= v_0 \cos\alpha \\ u^\theta(0) &= v_0 \sin\alpha \end{aligned}$$

where $\alpha \epsilon \Re$ and $v_0 \epsilon \Re$ is the length of the vector u.

Using this we obtain :-

$$\begin{aligned} A &= v_0 \cos\alpha \\ B &= v_0 \sin\alpha \end{aligned}$$

So, finally;

$$u^r(\lambda) = v_0(\cos\alpha\cos\lambda + \sin\alpha\sin\lambda) = v_0\cos(\alpha - \lambda)$$
$$u^\theta(\lambda) = v_0(\sin\alpha\cos\lambda - \cos\alpha\sin\lambda) = v_0\sin(\alpha - \lambda)$$

Now consider $\lambda = \frac{\pi}{2}$;

$$u^r\left(\frac{\pi}{2}\right) = v_0\sin\alpha = v_0\cos\left(\alpha - \frac{\pi}{2}\right)$$
$$u^\theta\left(\frac{\pi}{2}\right) = -v_0\cos\alpha = v_0\sin\left(\alpha - \frac{\pi}{2}\right)$$

Initially the vector u was making α angle with the $x-$axis. Now, one can clearly see in the above sets of pair of $eq^n s$ that, u makes $(\alpha - \lambda)$ angle with $r-$axis and r makes λ angle with $x-$axis. So, u makes α angle with $x-$axis. So it stays parallel to initial vector all throughout even at $\lambda = \frac{\pi}{2}$. This justifies the name parallel transport but this notion will become less trivial in the upcoming examples.

Example 2.2.2. Now consider $2-$sphere metric :-

$$ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\phi^2$$

where $a\epsilon\Re$ is the radius of the $2-$sphere.

Now, consider the following curve on it :-

$$C(\lambda) :- \ \theta(\lambda) = \frac{\pi}{2}$$
$$\phi(\lambda) = \lambda$$

Clearly, $C(\lambda)$ describes the equator. Now, let us compute the tangent vector fields to the curve.

$$t^\theta = \frac{d\theta}{d\lambda} = 0$$
$$t^\phi = \frac{d\theta}{d\lambda} = 1$$
$$So,\ t^\mu t_\mu = a^2 (constant)$$

So, the curve $C(\lambda)$ is affinely parametrized with affine parameter λ. Now, let us compute the non-vanishing components of the *Affine connection.*

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$
$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

Now, we would like to parallely transport a vector u along curve $C(\lambda)$. So, let us set up the parallel transport equations for this.

$$\frac{du^\mu}{d\lambda} + \Gamma^\mu_{\nu\rho} t^\nu u^\rho = 0$$
$$\Rightarrow \frac{du^\theta}{d\lambda} + \Gamma^\theta_{\phi\phi} t^\phi u^\phi = 0$$
$$\Rightarrow \frac{du^\theta}{d\lambda} = 0$$
$$Similarly,\ \frac{du^\phi}{d\lambda} + \Gamma^\phi_{\theta\phi} t^\theta u^\phi + \Gamma^\phi_{\phi\theta} t^\phi u^\theta = 0$$
$$\Rightarrow \frac{du^\phi}{d\lambda} = 0$$

So,

$$u^\theta(\lambda) = c_1$$
$$u^\phi = c_2$$

where $c_1, c_2 \epsilon \Re$.

Consider the given initial :-

$$\begin{aligned} u^\theta(0) &= 0 \\ u^\phi &= 1 \end{aligned}$$

Using this we obtain :-

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

Hence,

$$\begin{aligned} u^\theta(\lambda) &= 0 \\ u^\phi(\lambda) &= 1 \end{aligned}$$

So even at $\lambda = 2\pi$ this holds. So, parallely transporting a vector along the equator doesn't change anything and the parallely transported vector remains parallel to the initial vector at all times. Non-triviality is introduced in the next example.

Example 2.2.3. Here consider exactly the same scenario as before but the curve $C(\lambda)$ being redefined as :-

$$\begin{aligned} C(\lambda) :-\ \theta(\lambda) &= \alpha \\ \phi(\lambda) &= \lambda \end{aligned}$$

where $\alpha \epsilon \Re$. This curve clearly describes the latitudes. It can be easily checked like before that this is also affinely parametrized with affine parameter λ.

Now setting up the parallel transport equations we see :-

$$\begin{aligned} \frac{du^\theta}{d\lambda} - \sin\alpha\cos\alpha u^\phi &= 0 \\ And,\ \frac{du^\phi}{d\lambda} + \cot\alpha u^\theta &= 0 \end{aligned}$$

So,

$$\begin{aligned} \frac{d^2u^\theta}{d\lambda^2} - \sin\alpha\cos\alpha\frac{du^\phi}{d\lambda} &= 0 \\ \Rightarrow \frac{d^2u^\theta}{d\lambda^2} + \cos^2\alpha u^\theta &= 0 \\ Sol^n;\ u^\theta(\lambda) &= A\sin(\cos\alpha\lambda) + B\cos(\cos\alpha\lambda) \end{aligned}$$

where $A, B\epsilon\Re$. Now using the given initial conditions same as before we see :-

$$u^\theta(0) = B = 0$$

Hence,

$$u^\theta(\lambda) = A\sin(\cos\alpha\lambda)$$

This gives :-

$$u^\phi = \frac{A}{\sin\alpha}\cos(\cos\alpha\lambda)$$

Now using another initial condition we get :-

$$\begin{aligned} u^\phi(0) &= \frac{A}{\sin\alpha} = 1 \\ \Rightarrow A &= \sin\alpha \end{aligned}$$

So, we finally obtain :-

$$u^\theta(\lambda) = \sin\alpha(\cos\alpha\lambda)$$
$$u^\phi(\lambda) = \cos(\cos\alpha\lambda)$$

Consider $\lambda = 2\pi$:-

$$u^\theta(2\pi) = \sin\alpha(2\pi\cos\alpha)$$
$$u^\phi(2\pi) = \cos(2\pi\cos\alpha)$$

So, after parallel transport the initial and final vectors are identical. Now,

$$\begin{aligned} u^\mu_{,i}u_{\mu,f} &= a^2\sin^2\theta\cos(2\pi\cos\alpha) \\ u^\mu_{,i}u_{\mu,i} &= a^2 \\ u^\mu_{,f}u_{\mu,f} &= a^2\sin^2\alpha \end{aligned}$$

So, net change in final and initial norm of the given vector is :-

$$\Delta = a^2(1 - \sin^2\alpha) = a^2\cos^2\alpha$$

So, from the parallel transport equation it is clear that; given w at $p\epsilon M$ and curve γ; $\exists$ a unique sol^n of Eq^n 2.9 throughout the curve. In this way a curve-dependent identification can be made for vectors at $p\epsilon M$ and vectors at $p \neq q\epsilon M$ i.e., a curve-dependent identification of V_p and V_q, though they are still different. This curve-dependent identification adds a new mathematical structure on M known as a *connection.*

Now, we state a useful lemma involving parallel transport.

Lemma 2.2.1. *Let γ be a curve parametrized by $\lambda\epsilon\Re$ on M with tangent vector t. Let u, v be two vectors to parallely transported along γ. Then, their inner product is a constant iff a unique derivative operator ∇_a is chosen such that $\nabla_a g_{bc} = 0$; i.e.,*

$$\frac{d}{d\lambda}(g_{\mu\nu}u^\mu v^\nu) = 0 \;\Leftrightarrow\; \nabla_a g_{bc} = 0 \tag{2.10}$$

Proof.

$$\begin{aligned} \frac{d}{d\lambda}(g_{\mu\nu}u^\mu v^\nu) &= t^\alpha\nabla_\alpha(g_{\mu\nu}u^\mu v^\nu) \\ &= g_{\mu\nu}t^\alpha\nabla_\alpha(u^\mu v^\nu) + t^\alpha u^\mu v^\nu\nabla_\alpha g_{\mu\nu} \\ &= g_{\mu\nu}(u^\mu t^\alpha\nabla_\alpha v^\nu + v^\nu t^\alpha\nabla_\alpha u^\mu) + t^\alpha u^\mu v^\nu\nabla_\alpha g_{\mu\nu} \end{aligned}$$

Now the first two terms in the last eq^n vanishes owing to the parallel transport equation as u, v are parallely transported vectors along γ. So,

$$\frac{d}{d\lambda}(g_{\mu\nu}u^\mu v^\nu) = t^\alpha u^\mu v^\nu\nabla_\alpha g_{\mu\nu} \tag{2.11}$$

So, vanishing of the inner product of parallely transported vectors $\Leftrightarrow \nabla_\alpha g_{\mu\nu} = 0$.

Selecting such a derivative operator ∇_a for which $\nabla_a g_{bc} = 0$ provides us with a unique derivative operator. So, we obtain;

$$\frac{d}{d\lambda}(g_{\mu\nu}u^\mu v^\nu) = 0 \qquad \textbf{Proved}.$$

2.2.1 Affine Connection

Let us evaluate $\nabla_\alpha g_{\mu\nu} = 0$ to obtain the unique derivative operator ∇_a for which the former $cond^n$ holds.

$$\begin{aligned} \nabla_a g_{bc} &= \tilde{\nabla}_a g_{bc} - C^d_{ab}g_{dc} - C^d_{ac}g_{bd} = 0 \\ Now,\; \tilde{\nabla}_a g_{bc} &= C^d_{ab}g_{dc} + C^d_{ac}g_{bd} \;\cdots\cdots i) \\ \tilde{\nabla}_b g_{ac} &= C^d_{ba}g_{dc} + C^d_{bc}g_{ad} \;\cdots\cdots ii) \\ \tilde{\nabla}_c g_{ab} &= C^d_{cb}g_{da} + C^d_{ca}g_{db} \;\cdots\cdots iii) \end{aligned}$$

Now, *i*) + *ii*) − *iii*) along with symmetric nature of g_{ab} and lower indices of C^c_{ab} gives :-

$$
\begin{aligned}
\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab} &= C^d_{ac} g_{dc} + C^d_{ab} g_{dc} \\
\Rightarrow C^d_{ab} g_{dc} &= \frac{1}{2}(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab}) \\
\Rightarrow C_{cab} &= \frac{1}{2}(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab}) \\
\Rightarrow g^{dc} C_{cab} &= \frac{1}{2} g^{dc}(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab}) \\
\Rightarrow C^d_{ab} &= \frac{1}{2} g^{dc}(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab})
\end{aligned}
$$

Hence,

$$\Gamma^d_{ab} = \frac{1}{2} g^{dc}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}) \tag{2.12}$$

$$\Gamma^\phi_{\alpha\beta} = \frac{1}{2} g^{\phi\lambda}\left(\frac{\partial g_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial g_{\beta\lambda}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right) \tag{2.13}$$

In the above $Eq^n s$; Γ is known as the *Affine Connection* in accordance with the mathematical notion of *connection* presented earlier.

2.3 Riemann-Christoffel Curvature Tensor

We will show in this section that the *Riemann-Christoffel Curvature Tensor* captures the essence of intrinsic curvature of a manifold M.

Let $f \epsilon \mathcal{F}$ and w be a smooth vector field. Consider,

$$
\begin{aligned}
\nabla_a \nabla_b (f w_c) &= (\nabla_a f)(\nabla_b w_c) + f \nabla_a \nabla_b w_c + (\nabla_a w_c)(\nabla_b f) + w_c \nabla_a \nabla_b f \\
\nabla_b \nabla_a (f w_c) &= (\nabla_b f)(\nabla_a w_c) + f \nabla_b \nabla_a w_c + (\nabla_b w_c)(\nabla_a f) + w_c \nabla_b \nabla_a f
\end{aligned}
$$

So,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f w_c) = f(\nabla_a \nabla_b - \nabla_b \nabla_a) w_c$$

So, given $p \epsilon M$ we can observe that the map $(\nabla_a \nabla_b - \nabla_b \nabla_a)$ takes dual vectors at p to tensors of type $(0, 3)$ at p and its action is a type $(1, 3)$ tensor.

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) w_c = R_{abc}{}^d w_d \tag{2.14}$$

$R_{abc}{}^d$ is the *Riemann-Christoffel Curvature Tensor*.

Furthermore, for dual vector field w_b and vector field t^b we have :-

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(t^c w_c) = 0$$

This gives :-

$$
\begin{aligned}
&\Rightarrow \nabla_a \nabla_b (t^c w_c) - \nabla_b \nabla_a (t^c w_c) = 0 \\
&\Rightarrow t^c(\nabla_a \nabla_b w_c - \nabla_b \nabla_a w_c) + w_c(\nabla_a \nabla_b t^c - \nabla_b \nabla_a t^c) = 0 \\
&\Rightarrow t^c R_{abc}{}^d w_d + w_c(\nabla_a \nabla_b t^c - \nabla_b \nabla_a t^c) = 0 \\
&\Rightarrow w_c R_{abd}{}^c t^d + w_c(\nabla_a \nabla_b t^c - \nabla_b \nabla_a t^c) = 0
\end{aligned}
$$

So,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) t^c = -R_{abd}{}^c t^d \tag{2.15}$$

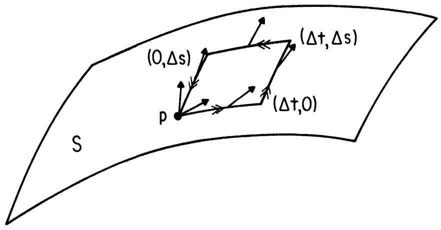

Figure 2.1: S is a two dimensional surface. (t, s) are coordinates of the surface. $p\epsilon S = (0, 0)$. v is a vector which is parallely transported along the closed loop shown as per the given direction starting from p back to p itself.

2.3.1 Path-dependence of Parallel Transport

Now, consider the following figure :-

We want to compute the change in v^a as it is parallely transported by taking its inner product with an arbitrary dual vector w_b and finding the change in this scalar. Let us set :-
δ_1 = variation in the inner product due to Δt variation along constant $s-$curve; $s = 0$.
δ_3 = variation in the inner product due to Δt variation along constant $s-$curve; $s = \Delta s$.
δ_2 = variation in the inner product due to Δs variation along constant $t-$curve; $t = 0$.
δ_4 = variation in the inner product due to Δs variation along constant $t-$curve; $t = \Delta t$.
T is tangent vector to constant $s-$curves.
S is tangent vector to constant $t-$curves.

So, considering midpoint derivatives which is accurate upto 2^{nd} order in displacement we get :-

$$\begin{aligned}\delta_1 &= \Delta t \left|\frac{\partial}{\partial t}(v^a w_a)\right|_{(\frac{\Delta t}{2},0)} = \Delta t \left|T^a \nabla_a (v^b w_b)\right|_{(\frac{\Delta t}{2},0)} \\ &= \Delta t \left|T^a v^b \nabla_a w_b\right|_{(\frac{\Delta t}{2},0)} \ (since\ T^a \nabla_a v^b = 0)\end{aligned}$$

Similarly, we have :-

$$\begin{aligned}\delta_3 &= \Delta t \left|T^a v^b \nabla_a w_b\right|_{(\frac{\Delta t}{2},\Delta s)} \\ So,\ \delta_1 + \delta_3 &= \Delta t[\left|T^a v^b \nabla_a w_b\right|_{(\frac{\Delta t}{2},0)} - \left|T^a v^b \nabla_a w_b\right|_{(\frac{\Delta t}{2},\Delta s)}] \\ And,\ \delta_2 + \delta_4 &= \Delta s[\cdots\cdots\cdots]\end{aligned}$$

Now, $\lim\limits_{\Delta s \to 0} \delta_1 + \delta_2 + \delta_3 + \delta_4 = 0$ (correct upto first order in Δt and Δs). So, parallel transport is independent to first order.

For second order change consider curve $t = \frac{\Delta t}{2}$. Consider, parallel transport of v^a and $T^a \nabla_a w_b$ along this curve from $(\frac{\Delta t}{2}, 0)$ to $\frac{\Delta t}{2}, \Delta s$. Clearly, since parallel transport is independent to first order, parallel transport of v^a won't contribute to the change, and change due to the parallel transport of $T^a \nabla_a w_b$ equals $\Delta s v^b S^c \nabla_c (T^a \nabla_a w_b)$. This gives;

$$\begin{aligned}\delta_1 + \delta_3 &= -\Delta t \Delta s v^b S^c \nabla_c (T^a \nabla_a w_b) \\ Similarly,\ \delta_2 + \delta_4 &= \Delta s \Delta t v^b T^c \nabla_c (S^a \nabla_a w_b) \\ Hence,\ \delta(v^b w_b) &= \Delta t \Delta s v^b [T^c \nabla_c (S^a \nabla_a w_b) - S^c \nabla_c (T^a \nabla_a w_b)]\end{aligned}$$

Now, since T and S are coordinate basis vectors their derivatives commute and hence we have;

$$\begin{aligned}\delta(v^b w_b) &= \Delta t \Delta s v^b [T^c S^a (\nabla_c \nabla_a w_b) - T^a S^c (\nabla_c \nabla_a w_b)] \\ &= \Delta t \Delta s v^b [T^a S^c (\nabla_a \nabla_c w_b) - T^a S^c (\nabla_c \nabla_a w_b)] \\ &= \Delta t \Delta s v^b T^a S^c [(\nabla_a \nabla_c - \nabla_c \nabla_a) w_b] \\ &= \Delta t \Delta s v^b T^a S^c R_{acb}{}^d w_d \qquad (2.16)\end{aligned}$$

Eq^n 2.16 shows that *Riemann-Christoffel Curvature Tensor* indeed measures the path-dependence of parallel transport. This shows that the parallel transport notion of curvature is linked to the *Riemann-Christoffel Curvature Tensor.*

2.3.2 Properties of Riemann-Christoffel Curvature Tensor

In this section we state some useful properties of *Riemann-Christoffel Curvature Tensor* without proofs.

$$R_{abc}{}^{d} = -R_{bac}{}^{d} \tag{2.17}$$

$$R_{abc}{}^{d} + R_{bca}{}^{d} + R_{cab}{}^{d} = 0 \tag{2.18}$$

$$R_{abcd} = R_{badc} = R_{cdab} \tag{2.19}$$

$$R_{abcd} = -R_{abdc} \tag{2.20}$$

2.4 Geodesics

Before defining *geodesics* with all the rigor we intuitively present the idea that geodesic is the shortest possible path or the straightest possible path between two points on the manifold.

Definition 2.4.1. Let γ be a curve parametrized by $\lambda\epsilon\Re$ on a manifold M with tangent vector t. γ is said to be a geodesic if t is parallely transported along itself, i.e.;

$$t^a\nabla_a t^b = 0 \; (\lambda \; is \; affine \; parameter) \tag{2.21}$$

$$t^a\nabla_a t^b = \alpha t^b \; (\lambda \; is \; non-affine \; parameter) \tag{2.22}$$

where, α is arbitrary function on γ. We will prove the difference between the above two *geodesic* equations in the penultimate section of this chapter.

In component form we have assuming λ to be the affine parameter :-

$$t^a(\partial_a t^b + \Gamma^b_{ac}t^c) = 0$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \tag{2.23}$$

Now we will prove that the curve of shortest distance between two points in a manifold indeed is indeed a geodesic.

Theorem 2.4.1. *Let γ be a curve parametrized by $\lambda\epsilon\Re$ on M with tangent vector t and let $p, q\epsilon M$. Furthermore, let γ be a geodesic connecting p and q. Then, γ is the shortest possible path between p and q lying on M.*

Proof. *Without loss of generality we assume λ is the affine parameter and since length of a curve is parameter-independent we assume that it has unit-speed parametrization, i.e.; $g_{\mu\nu}t^\mu t^\nu = 1$.*
Now, length between p and q is :-

$$l = \int_p^q [g_{\mu\nu}t^\mu t^\nu]^{1/2} d\lambda \tag{2.24}$$

From Calculus of Variations we know l is an extremum iff $\frac{d}{d\lambda}\left(\frac{\partial l}{\partial \dot{x}^\mu}\right) - \frac{\partial l}{\partial x^\mu} = 0$. Now, since $g_{\mu\nu}$ is independent of $\dot{x}^\mu$;

$$\frac{\partial l}{\partial \dot{x}^\alpha} = 2g_{\alpha\mu}\dot{x}^\mu = g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu$$

$$\frac{\partial l}{\partial x^\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu\dot{x}^\nu$$

$$\frac{d}{d\lambda}\left(\frac{\partial l}{\partial \dot{x}^\alpha}\right) = \frac{\partial g_{\alpha\nu}}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\alpha\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\mu}{d\lambda} + g_{\alpha\mu}\frac{d^2x^\mu}{d\lambda^2}$$

Now, $\frac{d}{d\lambda}\left(\frac{\partial l}{\partial \dot{x}^\alpha}\right) - \frac{\partial l}{\partial x^\alpha} = 0$ gives :-

$$\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\alpha\phi}\frac{d^2x^\phi}{d\lambda^2} = 0$$

Multiplying above by $\frac{1}{2}g^{\alpha\phi}$ *we get :-*

$$\frac{1}{2}g^{\alpha\phi}\left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}}+\frac{\partial g_{\alpha\nu}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right)\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}+\frac{d^2x^{\phi}}{d\lambda^2}=0$$

$$\Rightarrow\frac{d^2x^{\phi}}{d\lambda^2}+\Gamma^{\phi}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0 \tag{2.25}$$

Eq^n *2.25 is nothing but the geodesic equation.*
So, indeed γ *is the shortest path connecting* p *and* q *lying on* M*.* **Proved**.

2.4.1 Some Examples of Geodesics

Here we present some examples of geodesics.

Example 2.4.1. Here we will consider geodesics in a plane. Consider the polar flat $\Re^2$ metric :-

$$ds^2 = dr^2 + r^2 d\theta^2$$

Now let us set up the geodesic equation with affine parameter λ.

$$\frac{d^2x^{\phi}}{d\lambda^2}+\Gamma^{\phi}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0$$

$$\Rightarrow\frac{d^2x^{r}}{d\lambda^2}+\Gamma^{r}_{\theta\theta}\left(\frac{dx^{\theta}}{d\lambda}\right)^2=0$$

$$\Rightarrow\frac{d^2x^{r}}{d\lambda^2}-r\left(\frac{dx^{\theta}}{d\lambda}\right)^2=0$$

$$Similarly,\ \frac{d^2x^{\theta}}{d\lambda^2}+\Gamma^{\theta}_{\theta r}\left(\frac{dx^{\theta}}{d\lambda}\right)\left(\frac{dx^{r}}{d\lambda}\right)+\Gamma^{r}_{r\theta}\left(\frac{dx^{r}}{d\lambda}\right)\left(\frac{dx^{\theta}}{d\lambda}\right)=0$$

$$\Rightarrow\frac{d^2x^{\theta}}{d\lambda^2}+\frac{2}{r}\left(\frac{dx^{r}}{d\lambda}\right)\left(\frac{dx^{\theta}}{d\lambda}\right)=0$$

Consider,

$$\frac{d^2\theta}{d\lambda^2}+\frac{2}{r}\left(\frac{dr}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right)=0$$

$$Multiply\ by\ r^2;\ r^2\frac{d^2\theta}{d\lambda^2}+2r\left(\frac{dr}{d\lambda}\right)\left(\frac{d\theta}{d\lambda}\right)=0$$

$$Now,\ \frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right)=r^2\frac{d^2\theta}{d\lambda^2}+2r\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}$$

$$\Rightarrow\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right)=0$$

$$\Rightarrow r^2\frac{d\theta}{d\lambda}=c_1(constant)$$

$$\Rightarrow\frac{d\theta}{d\lambda}=\frac{c_1}{r^2}$$

$$Also,\ \frac{d^2r}{d\lambda^2}-r\left(\frac{d\theta}{d\lambda}\right)^2=0$$

$$\Rightarrow\frac{d^2r}{d\lambda^2}-r\frac{c_1^2}{r^4}=0$$

$$\Rightarrow\frac{d^2r}{d\lambda^2}=\frac{c_1^2}{r^3}$$

Now consider; unit-speed parametrization of the curve, i.e.; it is parametrized by the arc length $ds = d\lambda$. Hence;

$$\begin{aligned}
&\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 = 1\\
&\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = 1 - r^2\frac{c_1^2}{r^4} = 1 - \frac{c_1^2}{r^2}\\
&\Rightarrow \frac{dr}{d\lambda} = \pm\left(1 - \frac{c_1^2}{r^2}\right)^{\frac{1}{2}}\\
&\Rightarrow \frac{d^2r}{d\lambda^2} = \frac{c_1^2}{r^3}\\
So,\ indeed;\ &\frac{dr}{d\lambda} = \pm\left(1 - \frac{c_1^2}{r^2}\right)^{\frac{1}{2}}\\
&\Rightarrow \frac{r}{(r^2 - c_1^2)^{\frac{1}{2}}}dr = \pm d\lambda\ (with\ c_1^2 \leq r^2)\\
&\Rightarrow\ \pm(r^2 - c_1^2)^{\frac{1}{2}} = \lambda + \lambda_0\\
&\Rightarrow r^2 = c_1^2 + (\lambda + \lambda_0)^2\\
Now,\ &\frac{d\theta}{d\lambda} = \frac{c_1}{r^2} = \frac{c_1}{c_1^2 + (\lambda + \lambda_0)^2}\\
&\Rightarrow d\theta = \frac{c_1}{c_1^2 + (\lambda + \lambda_0)^2}d\lambda\\
&\Rightarrow \theta = \arctan\left(\frac{\lambda + \lambda_0}{c_1}\right) + \alpha\ (with\ \alpha\epsilon[0, 2\pi))
\end{aligned}$$

So,

$$\begin{aligned}
r^2(\lambda) &= c_1^2 + (\lambda + \lambda_0)^2\\
\theta(\lambda) &= \arctan\left(\frac{\lambda + \lambda_0}{c_1}\right)
\end{aligned}$$

So,

$$\begin{aligned}
&\tan(\theta - \alpha) = \frac{\lambda + \lambda_0}{c_1}\\
&\Rightarrow r^2 = c_1^2\sec^2(\theta - \alpha)\\
&\Rightarrow c_1^2 = r^2\cos^2(\theta - \alpha)\\
&\Rightarrow c_1 = \pm r\cos(\theta - \alpha)\\
&\Rightarrow c_1 = \pm[r\cos\theta\cos\alpha + r\sin\theta\sin\alpha] = \pm[x\cos\alpha + y\sin\alpha]
\end{aligned}$$

Now since, $c_1^2 \leq r^2$, where; $r\epsilon[0, \infty)$; so, $-r \leq c_1 \leq r$ and furthermore, $c_1\epsilon(-\infty, \infty)$.

Hence equation of geodesic obtained is :-

$$x\cos\alpha + y\sin\alpha = p$$

where $p\epsilon\Re$ is a constant. This is indeed the equation of a straight line in parametric form with p as the perpendicular distance of the line from the origin. Hence, geodesics of a plain are straight lines.

Example 2.4.2. Now we will consider geodesics in a 2−sphere. Consider the 2−sphere metric :-

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Now setting up the geodesic equation we see :-

$$\begin{aligned}
&\frac{d^2\theta}{d\lambda^2} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0\\
And,\ &\frac{d^2\phi}{d\lambda^2} + 2\cot\theta\left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) = 0
\end{aligned}$$

Now consider,

$$
\begin{aligned}
&\frac{d}{d\lambda}\left(f(\theta)\frac{d\phi}{d\lambda}\right) = 0 \\
\Rightarrow\ &\frac{df}{d\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} + f\frac{d^2\phi}{d\lambda^2} = 0 \\
\Rightarrow\ &\frac{d^2\phi}{d\lambda^2} + \frac{1}{f}\frac{df}{d\theta}\left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) = 0 \\
\Rightarrow\ &\frac{d^2\phi}{d\lambda^2} + \frac{dlnf}{d\theta}\left(\frac{d\theta}{d\lambda}\right)\left(\frac{d\phi}{d\lambda}\right) = 0
\end{aligned}
$$

So, let :-

$$
\begin{aligned}
&\frac{dlnf}{d\theta} = 2\cot\theta \\
\Rightarrow\ &f(\theta) = \sin^2\theta
\end{aligned}
$$

So, we get :-

$$
\begin{aligned}
&\frac{d}{d\lambda}\left(\sin^2\theta\frac{d\phi}{d\lambda}\right) = 0 \\
\Rightarrow\ &\frac{d\phi}{d\lambda} = \frac{c_1}{\sin^2\theta} \\
\Rightarrow\ &\frac{d^2\theta}{d\lambda^2} = \sin\theta\cos\theta\frac{c_1^2}{\sin^4\theta} = c_1^2\frac{\cos\theta}{\sin^3\theta}
\end{aligned}
$$

Now consider; unit-speed parametrization of the curve, i.e.; it is parametrized by the arc length $ds = d\lambda$. Hence;

$$
\begin{aligned}
\left(\frac{d\theta}{d\lambda}\right)^2 &= 1 - \sin^\theta\left(\frac{d\phi}{d\lambda}\right)^2 \\
&= 1 - \sin^2\theta\frac{c_1^2}{\sin^4\theta} = 1 - \frac{c_1^2}{\sin^2\theta}
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{d\theta}{d\lambda} = \pm\left(1 - \frac{c_1^2}{\sin^2\theta}\right)^{\frac{1}{2}} \\
\Rightarrow\ &\frac{d^2\theta}{d\lambda^2} = c_1^2\frac{\cos\theta}{\sin^3\theta} \\
So,\ indeed;\ &\frac{d\theta}{d\lambda} = \pm\left(1 - \frac{c_1^2}{\sin^2\theta}\right)^{\frac{1}{2}} \\
\Rightarrow\ &\frac{\sin\theta}{(\sin^2\theta - c_1^2)^{\frac{1}{2}}}d\theta = \pm d\lambda \\
\Rightarrow\ &\cos\theta = (1 - c_1^2)^{\frac{1}{2}}\sin(\lambda + \lambda_0) \\
Now,\ &\sin^2\theta = 1 - \cos^2\theta = 1 - (1 - c_1^2)\sin^2(\lambda + \lambda_0) \\
&= c_1^2\sin^2(\lambda + \lambda_0) + \cos^2(\lambda + \lambda_0) \\
\Rightarrow\ &d\phi = \frac{c_1}{c_1^2\sin^2(\lambda + \lambda_0) + \cos^2(\lambda + \lambda_0)}d\lambda \\
\Rightarrow\ &\phi = c_1\tanh^{-1}((-1 + a^2)^{\frac{1}{2}}\tan(\lambda + \lambda_0))\frac{1}{(-1 + a^2)^{\frac{1}{2}}} + \phi_0
\end{aligned}
$$

where $a^2 = 1 - c_1^2$, this implies; $(-1 + a^2)^{\frac{1}{2}} = \iota c_1$.

Now,

$$
\begin{aligned}
&\tanh\iota x = \frac{e^{\iota x} - e^{-\iota x}}{e^{\iota x} + e^{-\iota x}} = \iota\frac{\sin x}{\cos x} = \iota\tan x \\
\Rightarrow\ &\tanh^{-1}\iota x = \iota\arctan x
\end{aligned}
$$

Now,

$$\begin{aligned}\phi &= c_1 \tanh^{-1}(\iota c_1 \tan(\lambda+\lambda_0))\frac{1}{\iota c_1} + \phi_0 \\ \phi - \phi_0 &= \arctan(c_1 \tan(\lambda+\lambda_0))\end{aligned}$$

Now taking $s = 0$ at $\theta = \frac{\pi}{2}$ we get; $\lambda = 0$. So,

$$\begin{aligned}\cos\theta(\lambda) &= \pm a \sin\lambda \\ \tan(\phi(\lambda) - \phi_0) &= c_1 \tan\lambda\end{aligned}$$

Now, to see what these geodesic equations mean define planes containing $x-$axis in $\Re^3$ as :-

$$z = my$$

where $m\epsilon\Re$.

Now, eq^n of sphere is $x^2 + y^2 + z^2 = 1$. Intersection of this with the above plains gives great circles containing the $x-$axis. So,

$$\begin{aligned}& x^2 + y^2 + m^2 y^2 = 1 \\ \Rightarrow\ & \sin^2\theta\cos^2\phi + (1+m^2)\sin^2\theta\sin^2\phi = 1 \\ \Rightarrow\ & \sin^2\theta(\cos^2\phi + \sin^2\phi + m^2\sin^{\phi}) = 1 \\ \Rightarrow\ & \pm\sin\phi = \left(\frac{1}{\sin^2\theta - 1}\right)^{\frac{1}{2}} = \cot\theta\end{aligned}$$

So, $\pm\sin\phi = \cot\theta$ is the equation of great circles including intersection of $x-$axis with $2-$sphere. Now, let $\phi_0 = 0$ $(\phi = 0 \rightarrow x - axis;\ s = 0\ at\ equator)$. So,

$$\begin{aligned}\cos\theta &= \pm a\sin\lambda \\ \tan\phi &= c_1 \tan\lambda\end{aligned}$$

Now,

$$\begin{aligned}\cos^2\theta &= a^2 \frac{\tan^2\lambda}{1+\tan^2\lambda} \\ \Rightarrow \frac{1}{1+\tan^2\theta} &= a^2 \frac{\tan^2\theta}{c_1^2 + \tan^2\phi} \\ \Rightarrow \frac{\cot^2\theta}{1+\cot^2\theta} &= a^2 \frac{\sin^2\phi}{c_1^2(1-\sin^2\phi) + \sin^2\phi} \\ &= \frac{(1-c_1^2)\sin^2\phi}{c_1^2 + (1-c_1^2)\sin^2\phi} \\ &= \frac{\frac{1-c_1^2}{c_1^2}\sin^2\phi}{1 + \frac{1-c_1^2}{c_1^2}\sin^2\phi} \\ \Rightarrow \cot^2\theta &= \frac{1-c_1^2}{c_1^2}\sin^2\phi \\ \Rightarrow \cot\theta &= \pm\left(\frac{1-c_1^2}{c_1^2}\right)^{\frac{1}{2}}\sin\phi = \pm m\sin\phi\end{aligned}$$

So, identifying;

$$m = \left(\frac{1-c_1^2}{c_1^2}\right)^{\frac{1}{2}}$$

we conclude that geodesics on $2-$sphere are indeed parts of a great circle, i.e; arcs.

2.4.2 Category of geodesics based on Signature of their Tangent Vector

Let γ be a curve parametrized by $\lambda \epsilon \Re$ on a manifold M with tangent vector t. Furthermore, let γ be a geodesic. Now, if :-
i) $t^\mu t_\mu < 0$, then γ is said to be a timelike geodesic.
ii) $t^\mu t_\mu = 0$, then γ is said to be a nulllike geodesic.
iii) $t^\mu t_\mu > 0$, then γ is said to be a spacelike geodesic.

2.4.3 First Integral of Geodesic Equation

We state the first integral of the *geodesic equation* by stating a useful lemma.

Lemma 2.4.2. *Let γ be a curve affinely parametrized by $\lambda \epsilon \Re$ on a manifold M with tangent vector t. Furthermore, let γ be a geodesic. Then,*

$$\frac{d}{d\lambda}(g_{\mu\nu}t^\mu t^\nu) = 0 \tag{2.26}$$

Proof. *Consider,*

$$\begin{aligned}\frac{d}{d\lambda}(g_{\mu\nu}t^\mu t^\nu) &= t^\alpha \nabla_\alpha (g_{\mu\nu}t^\mu t^\nu) \\ &= t^\alpha t^\mu t^\nu \nabla_\alpha g_{\mu\nu} + g_{\mu\nu}t^\mu t^\alpha \nabla_\alpha t^\nu + g_{\mu\nu}t^\nu t^\alpha \nabla_\alpha t^\mu\end{aligned}$$

In the last eq^n the first term vanishes due to the uniqueness of the derivative operator and the rest two terms vanish owing to the geodesic equation.
So, we get;

$$\frac{d}{d\lambda}(g_{\mu\nu}t^\mu t^\nu) = 0 \qquad \textbf{Proved}.$$

2.4.4 Geodesic Deviation

Let $\gamma_s(t)$ denote a smooth one-parameter family of geodesics, i.e.; for each $s\epsilon\Re$, the curve γ_s is a geodesic (parameterized by affine parameter t); and the map $(t,s) \to \gamma_s(t)$ is bijective and smooth, and has smooth inverse. Let Σ denote the two dimensional submanifold spanned by the curves $\gamma_s(t)$. We may choose (s,t) as coordinates of Σ. Now, let :-
T^a is tangent vector to constant $s-$curves.
X^a is tangent vector to constant $t-$curves.
We also have the following observations since T^a, X^a are coordinates curves their derivatives commute so (which will be proved in detail later in *geodesic deviation revisited*) :-

$$T^a \nabla_a X^b = X^a \nabla_a T^b \tag{2.27}$$

$$Also,\ T^a \nabla_a T^b = 0 \tag{2.28}$$

X^a is known as the *deviation vector* as it measures how far the geodesics are from each other. Now, let us define velocity vector v^a which gives the rate of change along a geodesic of the displacement to an infinitesimally nearby geodesic.

$$v^a = T^b \nabla_b X^a \tag{2.29}$$

Similarly, we may interpret :-

$$a^a = T^c \nabla_c v^a = T^c \nabla_c (T^b \nabla_b X^a) \tag{2.30}$$

as the relative acceleration of an infinitesimally nearby geodesic in the family. Now,

$$\begin{aligned}a^a &= T^c \nabla_c (T^b \nabla_b X^a) \\ &= T^c \nabla_c (X^b \nabla_b T^a) \\ &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a \\ &= (X^c \nabla_c T^b)(\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a - R_{cbd}{}^a X^b T^c T^d \\ &= (X^c \nabla_c)(T^b \nabla_b T^a) + X^b \nabla_b (T^c \nabla_c T^a) - (X^b \nabla_b T^c)(\nabla_c T^a) - R_{cbd}{}^a X^b T^c T^d \\ &= -R_{cbd}{}^a X^b T^c T^d\end{aligned} \tag{2.31}$$

Eq^n 2.31 is known as the *geodesic deviation equation.*
So, initially parallel geodesics, i.e.; $v^a = 0$ will remain parallel if along γ_s; $a^a = 0 \Leftrightarrow R_{cbd}{}^a = 0$. This shows that the parallelness of geodesics' notion of curvature is indeed linked to the *Riemann-Christoffel Curvature Tensor.*

Thus, the *Riemann-Christoffel Curvature Tensor* captures the meaning of curvature on a manifold.

2.5 The two versions of Geodesic Equations

Here we will show that for affinely parametrized and non-affinely parametrized curves, the *geodesic* equations are given by $eq^n s$ 2.21 and 2.22 respectively.

Consider affinely parametrized *geodesic* with tangent vector field t such that $t^\mu t_\mu = const$. So, now;

$$t^\beta \nabla_\beta (t^\alpha t_\alpha) = 2t^\beta t^\alpha \nabla_\beta t_\alpha = 2t^\alpha t^\beta \nabla_\beta t_\alpha$$

But since $t^\alpha t_\alpha$ is a constant by definition, we have;

$$\begin{aligned} & t^\beta \nabla_\beta (t^\alpha t_\alpha) = 0 \\ \Rightarrow\ & 2t^\alpha t^\beta \nabla_\beta t_\alpha = 0 \\ \Rightarrow\ & 2t_\alpha t^\alpha t^\beta \nabla_\beta t_\alpha = 0 \ (multiplying\ by\ t_\alpha) \\ \Rightarrow\ & 2const\ t^\beta \nabla_\beta t_\alpha = 0 \end{aligned}$$

So, we finally get from above :-

$$t^\beta \nabla_\beta t_\alpha = 0 \tag{2.32}$$

Now, consider non-affinely parametrized *geodesic* with parameter τ and tangent vector field t. So,

$$t^\beta \nabla_\beta (t^\alpha t_\alpha) = 2t^\beta t^\alpha \nabla_\beta t_\alpha = 2t^\alpha t^\beta \nabla_\beta t_\alpha$$

But, $t^\alpha t_\alpha = f(\tau)$ where f is a non-zero and non-constant smooth function of τ by definition. So,

$$\begin{aligned} & t^\beta \nabla_\beta (f(\tau)) = \frac{df}{d\tau} \\ \Rightarrow\ & 2t^\alpha t^\beta \nabla_\beta t_\alpha = \frac{df}{d\tau} \\ \Rightarrow\ & 2t_\alpha t^\alpha t^\beta \nabla_\beta t_\alpha = t_\alpha \frac{df}{d\tau} \ (multiplying\ by\ t_\alpha) \\ \Rightarrow\ & 2f(\tau) t^\beta \nabla_\beta t_\alpha = t_\alpha \frac{df}{d\tau} \\ \Rightarrow\ & t^\beta \nabla_\beta t_\alpha = \frac{1}{2f(\tau)} \frac{df}{d\tau} t_\alpha \end{aligned}$$

So, denoting $g(\tau) = \frac{1}{2f(\tau)} \frac{df}{d\tau}$ which is indeed an arbitrary smooth function of τ due to arbitrariness of $f(\tau)$ we have;

$$t^\beta \nabla_\beta t_\alpha = g(\tau) t_\alpha \tag{2.33}$$

$$t^\beta \nabla_\beta t_\alpha\ \alpha\ t_\alpha \tag{2.34}$$

2.6 Physical Motivations

In this section we present the physical motivations behind constructing the *geodesic* equation and parallel transport equation as they are.

2.6.1 Geodesic Equation

Consider first that the *geodesic* would be a curve where a vector defined kind of an acceleration vector in a covariant sense would vanish (this is in analogy to particles moving in straight lines having zero accelerations).

Now consider timelike curves (particle trajectories). Let it be $x^\mu(\tau)$. So, tangent vector $u^\mu = \frac{dx^\mu}{d\tau}$ would be velocity.

Now, $u^\mu{}_{;\nu}$ equals rate of change of velocity (i.e.; acceleration). Furthermore, $u^\mu{}_{;\nu}u^\nu$ equals the projection of acceleration along timelike direction (since scalar product with a vector is nothing but the component projection along the vector's direction).

So, following previous analogies we demand component of acceleration along timelike direction (i.e.; along particle trajectories) to vanish. So,

$$u^\mu{}_{;\nu}u^\nu = 0 \tag{2.35}$$

which is nothing but the *geodesic* equation.

2.6.2 Parallel Transport Equation

From previous arguments we see that given a curve $C(\lambda)$ with tangent vector field t and affine parameter λ; a vector u to be parallely transported along $C(\lambda)$, consider; $u^\alpha{}_{;\beta}$ which equals the rate of change of the vector in a covariant sense.

Furthermore, $u^\alpha{}_{;\beta}t^\beta$ equals the rate of change of vector along the curve as it is projected along the tangent vector.

Now, from Euclidean geometry parallel transport of a vector along a curve heuristically speaking "doesn't change anything related to the vector". So, motivated by this we find;

$$u^\alpha{}_{;\beta}t^\beta = 0 \tag{2.36}$$

which is nothing but the parallel transport equation.

From this it is straightaway clear that since *geodesic* is a curve whose tangent vector is parallely propagated along itself, we have;

$$t^\alpha{}_{;\beta}t^\beta = 0 \tag{2.37}$$

Chapter 3

Lie Differentiation

3.1 Lie Derivative

In previous chapter, given a manifold M, we saw *Covariant derivative* was defined by introducing a rule to transport a tensor from a point $q\epsilon M$ to a neighbouring point $p\epsilon M$, at which the derivative was to be evaluated. This rule involved the introduction of a new structure on the manifold, the *connection.* In this chapter we define another type of derivative, more natural one, the *Lie derivative* without introducing any additional structure.

Now, let γ be a curve on a manifold M parametrized by $\lambda\epsilon\Re$; with $u^\alpha = \frac{dx^\alpha}{d\lambda}$ as the tangent vector. Now let A^μ be a smooth vector field field defined in the *nbd* of $p\epsilon M$ on γ. Now, consider another point $q\epsilon M$ in the same *nbd* of p. Furthermore, let coordinates of p be x^α while q be $x^{'\alpha} = x^\alpha + dx^\alpha = x^\alpha + u^\alpha d\lambda$. Now, under this infinitesimal coordinate transformation; the vector A^μ becomes;

$$\begin{aligned} A^{'\alpha} &= \frac{\partial x^{'\alpha}}{\partial x^\mu} A^\mu \\ &= \left(\delta^\alpha_\mu + \frac{\partial u^\alpha}{\partial x^\mu} d\lambda\right) A^\mu \ (by\ transformation\ eq^n) \\ &= A^\alpha + \frac{\partial u^\alpha}{\partial x^\mu} A^\mu d\lambda \end{aligned}$$

So, we get :-

$$A^{'\alpha}(q) = A^\alpha(p) + \frac{\partial u^\alpha}{\partial x^\mu} A^\mu(p) d\lambda \tag{3.1}$$

Now, let us find the value of the original vector A^μ at q; using Taylor's theorem we see :-

$$\begin{aligned} A^\alpha(q) &= A^\alpha(x + dx) \\ &= A^\alpha(x) + \frac{\partial A^\alpha}{\partial x^\mu} dx^\mu \end{aligned}$$

So, we get :-

$$A^\alpha(q) = A^\alpha(p) + \frac{\partial A^\alpha}{\partial x^\mu} u^\mu d\lambda \tag{3.2}$$

Now let us define *lie derivative* of A^μ along γ :-

Definition 3.1.1. *lie derivative* $\pounds(A)$ *of* A *along* γ :-

$$\pounds_u A^\alpha(p) = \frac{A^\alpha(q) - A^{'\alpha}(q)}{d\lambda} \tag{3.3}$$

$$= \frac{\partial A^\alpha(p)}{\partial x^\mu} u^\mu - \frac{\partial u^\alpha}{\partial x^\mu} A^\mu(p) \tag{3.4}$$

Now, recall;

$$\begin{aligned} &\nabla_\mu A^\alpha = \partial_\mu A^\alpha + \Gamma^\alpha_{\mu\nu} A^\nu \\ &Multiplying\ by\ u^\mu\ ;\ \Rightarrow \partial_\mu A^\alpha u^\mu = u^\mu \nabla_\mu A^\alpha - \Gamma^\alpha_{\mu\nu} u^\mu A^\nu \ \cdots\cdots\ i) \\ &Also,\ \nabla_\mu u^\alpha = \partial_\mu u^\alpha + \Gamma^\alpha_{\mu\nu} u^\nu \\ &Multiplying\ by\ A^\mu\ ;\ \Rightarrow \partial_\mu u^\alpha A^\mu = A^\mu \nabla_\mu u^\alpha - \Gamma^\alpha_{\mu\nu} u^\nu A^\mu \ \cdots\cdots\ ii) \end{aligned}$$

By symmetry of lower indices of $\Gamma^{\alpha}_{\mu\nu}$; $i) \; - \; ii)$ gives :-

$$\pounds_u A^{\alpha}(p) = u^{\mu}\nabla_{\mu}A^{\alpha}(p) - A^{\mu}(p)\nabla_{\mu}u^{\alpha} \tag{3.5}$$

So, this shows for $f \epsilon \mathcal{F}$:-

$$\pounds_u f = \frac{df}{d\lambda} = u(f) \tag{3.6}$$

Furthermore, recall :-

$$[v, w]^b = v^a\nabla_a w^b - w^a\nabla_a v^b$$
$$So,\; [u, A]^{\alpha} = u^{\mu}\nabla_{\mu}A^{\alpha} - A^{\mu}\nabla_{\mu}u^{\alpha}$$

Hence, we obtain :-

$$\pounds_u A^{\alpha}(p) = [u, A]^{\alpha}(p) \tag{3.7}$$

3.2 Lie Transport along a Curve

Let γ be a curve on a manifold M parametrized by $\lambda\epsilon\Re$; with $u^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$ as the tangent vector. Consider a coordinate system such that $x^1, x^2, x^3 - constant$ on γ and only $x^0 \equiv \lambda$ varies on γ. So,

$$u^{\alpha} = \frac{dx^{\alpha}}{d\lambda} = \delta^{\alpha}_0 \tag{3.8}$$

Now,

Definition 3.2.1. A smooth vector field A is said to be *lie transported* along γ if :-

$$\pounds_u A^{\alpha} = 0 \tag{3.9}$$

So, this becomes in the chosen coordinate system :-

$$\pounds_u A^{\alpha} = \partial_{\mu}A^{\alpha}\delta^{\mu}_0 - \partial_{\mu}\delta^{\alpha}_0 A^{\mu} = \frac{\partial A^{\alpha}}{\partial x^0} = 0 \tag{3.10}$$

Similar result for arbitrary smooth tensor fields. So, if a tensor is lie transported along γ, i.e.; $\pounds_u T^{\alpha\cdots\cdots}_{\beta\cdots\cdots} = 0$ with tangent vector u^{α}; then a coordinate system can be constructed such that $u^{\alpha} = \delta^{\alpha}_0$ and $\frac{\partial T^{\alpha\cdots\cdots}_{\beta\cdots\cdots}}{\partial x^0} = 0$.

Conversely, if in a given coordinate system the components of if $T^{\alpha\cdots\cdots}_{\beta\cdots\cdots}$ in independent of x^0 then $\pounds_u T^{\alpha\cdots\cdots}_{\beta\cdots\cdots} = 0$.

3.3 Geodesic Deviation revisited

Consider the following figure :-

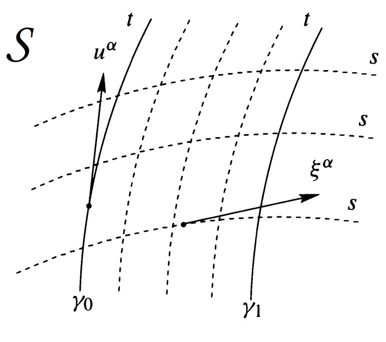

Figure 3.1: S is a two dimensional surface. (t, s) are coordinates of the surface. Constant $s-$curves are geodesics e.g.; γ_0 and γ_1 with $s = 0$ and $s = 1$ respectively. Geodesics are described with relations $x^{\alpha}(t, s)$, in which s serves to specify which geodesic and t is an affine parameter along the specified geodesic. The vector field $u^{\alpha} = \frac{dx^{\alpha}}{dt}$ is tangent to the geodesics, and it satisfies the equation $u^{\beta}\nabla_{\beta}u^{\alpha} = 0$

Now,

$$u^\alpha = \frac{\partial x^\alpha}{\partial t} \ (const.\ s - curve\ tangent)$$
$$\xi^\alpha = \frac{\partial x^\alpha}{\partial s} \ (const.\ t - curve\ tangent)$$

So,

$$\frac{\partial u^\alpha}{\partial s} = \frac{\partial^2 x^\alpha}{\partial s \partial t} = \frac{\partial^2 x^\alpha}{\partial t \partial s} = \frac{\partial \xi^\alpha}{\partial t} \tag{3.11}$$

Now, since $\pounds_u \xi^\alpha = [u, \xi]^\alpha$ and $\pounds_\xi u^\alpha = [\xi, u]^\alpha$ and futhermore, since; u and ξ are tangent vectors to coordinate curves, their commutator vanishes. This gives :-

$$\pounds_u \xi^\alpha = [u, \xi]^\alpha = [\xi, u]^\alpha = \pounds_\xi u^\alpha = 0 \tag{3.12}$$
$$So,\ u^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta u^\alpha \tag{3.13}$$

Eq^n 3.13 is equivalent Eq^n 2.27. This is how we derived it here.

Now,

$$\begin{aligned}
\frac{d}{dt}(\xi^\alpha u_\alpha) &= u^\beta \nabla_\beta (\xi^\alpha u_\alpha) \\
&= u_\alpha u^\beta \nabla_\beta \xi^\alpha + \xi^\alpha (u^\beta \nabla_\beta u_\alpha) \\
&= u_\alpha (u^\beta \nabla_\beta \xi^\alpha) \ (owing\ to\ geodesic\ equation) \\
&= u_\alpha (\xi^\beta \nabla_\beta u^\alpha) \ (owing\ to\ Eq^n\ 3.13) \\
&= \frac{1}{2} \xi^\beta \nabla_\beta (u^\alpha u_\alpha) \ (by\ Leibnitz\ rule)
\end{aligned}$$

The last line equals zero as $u^\alpha u_\alpha$ is a constant since u is tangent to affinely parametrized geodesic γ. So, we get :-

$$\frac{d}{dt}(\xi^\alpha u_\alpha) = 0 \tag{3.14}$$

Hence, $\xi^\alpha u_\alpha$ is constant along γ. Since, $\exists$ a *gauge freedom* in ξ owing to the arbitrariness in its direction it can be picked in such a way that :-

$$\xi^\alpha u_\alpha = 0 \tag{3.15}$$

This does away with the *gauge freedom* of ξ.

3.4 Killing Vector Fields

Definition 3.4.1. If in a given coordinate, $g_{\mu\nu}$ is independent of x^0 then, from previous discussion we observe; $\pounds_\xi g_{\mu\nu} = 0$, where; $\xi^\alpha = \delta^\alpha_0$. Then, ξ is known as a *killing vector field* of the geometry.

Now, from the defining eq^n of *lie derivative* it can be shown :-

$$\pounds_\xi u_\alpha = \xi^\mu \nabla_\mu u_\alpha + u_\mu \nabla_\alpha \xi^\mu \tag{3.16}$$

So, we get from induction :-

$$\begin{aligned}
\pounds_\xi g_{\alpha\beta} &= \xi^\mu \nabla_\mu g_{\alpha\beta} + g^\alpha \mu \nabla_\beta \xi^\mu + g_{\mu\beta} \nabla_\alpha \xi^\mu = 0 \\
But,\ \nabla_\mu g_{\alpha\beta} &= 0 \\
So,\ g_{\alpha\mu} \nabla_\beta \xi^\mu &+ g_{\mu\beta} \nabla_\alpha \xi^\mu = 0
\end{aligned}$$

So, we obtain what is known as the *Killing's Equation* :-

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \tag{3.17}$$
$$\Rightarrow\ \nabla_{[\alpha} \xi_{\beta]} = 0 \tag{3.18}$$

Sol^n of *Killing's Equation* gives all the *killing vector fields* of the geometry.

3.4.1 Relation to Symmetry and Constants of Motion

Let γ be a curve on a manifold M affinely parametrized by $\lambda\epsilon\Re$; with $u^\alpha = \frac{dx^\alpha}{d\lambda}$ as the tangent vector and furthermore, let ξ be a *killing vector field* of the geometry. Then,

$$u^\beta\nabla_\beta(u^\alpha\xi_\alpha) = (u^\beta\nabla_\beta u^\alpha)\xi_\alpha + u^\beta(u^\alpha\nabla_\beta\xi_\alpha)$$

The first term in the above eq^n vanishes owing to the *geodesic equation* and the second term owing to the fact that $u^\beta u^\alpha$ is symmetric and $\nabla_\beta\xi_\alpha$ is antisymmetric in the indices. So, we get :-

$$u^\beta\nabla_\beta(u^\alpha\xi_\alpha) = 0 \tag{3.19}$$

So, it is clear from above that $u^\alpha\xi_\alpha$ is constant along γ. In case of timelike geodesics this will be a *constant of motion.*

3.4.2 Another First Integral of Geodesic Equation

We state another first integral of the *geodesic equation* by stating a useful lemma.

Lemma 3.4.1. *Let γ be a curve affinely parametrized by $\lambda\epsilon\Re$ on a manifold M with tangent vector t. Furthermore, let γ be a geodesic. Now, let; ξ be a killing vector field of the geometry. Then,*

$$\frac{d}{d\lambda}(g_{\mu\nu}\xi^\mu t^\nu) = 0 \tag{3.20}$$

Proof. *Consider,*

$$\begin{aligned}\frac{d}{d\lambda}(g_{\mu\nu}\xi^\mu t^\nu) &= t^\alpha\nabla_\alpha(g_{\mu\nu}\xi^\mu t^\nu)\\ &= t^\alpha\xi^\mu t^\nu\nabla_\alpha g_{\mu\nu} + g_{\mu\nu}\xi^\mu t^\alpha\nabla_\alpha t^\nu + g_{\mu\nu}t^\nu t^\alpha\nabla_\alpha\xi^\mu\\ &= t^\alpha\xi^\mu t^\nu\nabla_\alpha g_{\mu\nu} + g_{\mu\nu}\xi^\mu t^\alpha\nabla_\alpha t^\nu + t^\alpha t^\nu\nabla_\alpha\xi_\nu\end{aligned}$$

In the last eq^n the first term vanishes owing to the uniqueness of the derivative operator; the second term vanishes owing to the geodesic equation and the last term vanishes since $t^\nu t^\alpha$ is symmetric and $\nabla_\alpha\xi_\nu$ is antisymmetric in its indices owing to the Killing's Equation.

So, we get;

$$\frac{d}{d\lambda}(g_{\mu\nu}\xi^\mu t^\nu) = 0 \qquad \textbf{Proved}.$$

3.4.3 Killing vector Lemmas

Consider, starting from eq^n 2.14 :-

$$\begin{aligned}\nabla_a\nabla_b\xi_c &= R_{abc}{}^d\xi_d + \nabla_b\nabla_a\xi_c\\ Killing's\ Equation,\ \nabla_b\xi_c + \nabla_c\xi_b &= 0\\ Now,\ \nabla_a\nabla_b\xi_c + \nabla_a\nabla_c\xi_b &= 0\\ From\ 1^{st}\ line,\ R_{abc}{}^d\xi_d + \nabla_b\nabla_a\xi_c + \nabla_a\nabla_c\xi_b &= 0\ \cdots\cdots i)\\ Cyclic\ Permutation,\ R_{bca}{}^d\xi_d + \nabla_c\nabla_b\xi_a + \nabla_b\nabla_a\xi_c &= 0\ \cdots\cdots ii)\\ R_{cab}{}^d\xi_d + \nabla_a\nabla_c\xi_b + \nabla_c\nabla_b\xi_a &= 0\ \cdots\cdots iii)\end{aligned}$$

Now, $i)\ +\ ii)\ -\ iii)$ gives :-

$$\begin{aligned}(R_{abc}{}^d + R_{bca}{}^d + R_{cab}{}^d)\xi_d + 2\nabla_b\nabla_a\xi_c &= 0\\ \Rightarrow \nabla_b\nabla_a\xi_c &= \frac{1}{2}(R_{cab}{}^d - R_{bca}{}^d - R_{abc}{}^d)\xi_d\\ &= \frac{1}{2}(R_{cab}{}^d + R_{cab}{}^d)\ (by\ Eq^n\ 2.18)\\ &= R_{cab}{}^d)\xi_d\\ \Rightarrow \nabla_b\nabla_a\xi_c &= -R_{acb}{}^d\xi_d\ (by\ Eq^n\ 2.17)\end{aligned}$$

So, we obtain :-

$$\nabla_a \nabla_b \xi_c = -R_{bca}{}^d \xi_d \tag{3.21}$$

Furthermore;

$$\nabla_a \nabla_b \xi^c = R^c{}_{bad} \xi^d \tag{3.22}$$

$Eq^n s$ 3.21 and 3.22 are known as the *Killing Vector Lemmas*.

3.4.4 Maximally Symmetric Spaces

Consider *Killing's Equation*; $\nabla_{[\alpha}\xi_{\beta]} = 0$. In d dimensional manifold M, $\exists$ $\frac{d(d-1)}{2}$ pairwise combinations of α, β and d same αs or βs to the *Killing's Equation* giving maximum $\frac{d(d-1)}{2} + d = \frac{d(d+1)}{2}$ $sol^n s$ to the *Killing's Equation*. Furthermore, consider; at any point $p \epsilon M$; $\xi_\mu(p)$ if is given, then; $\nabla_\nu \xi_\rho + \nabla_\rho \xi_\nu = 0$ can be uniquely solved to determine *linearly independent killing vectors* at p.

So, in $(\xi_\mu, \nabla_\nu \xi_\rho)$; maximum free indices equals $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$. So, $\exists$ maximum $\frac{d(d+1)}{2}$ *linearly independent killing vectors* in d dimensional manifold M.

The above argument is just a sketchy explanation not a complete rigorous proof of the discussed fact. Now, let us define what we mean by *Maximally Symmetric Spaces*.

Definition 3.4.2. Let M be a d dimensional manifold. If M admits $\frac{d(d+1)}{2}$ i.e.; maximum *linearly independent killing vectors* of the given geometry then M is called a *Maximally Symmetric Space*.

Chapter 4

Hypersurfaces

In this chapter we will deal with *hypersurfaces* and their relations to *killing vector fields* of the given geometry.

4.1 Normal Vector Fields

Let M be a n dimensional manifold with $\mathcal{N}$ as a $n-1$ submanifold (a *hypersurface*) with its tangent plane as $n-1$ dimensional subspace of the tangent plane of M. The family of *hypersurfaces* Σ is described by coordinate restriction as :-

$$\Phi(x^{\alpha}) = k\epsilon\Re \ (description\ of\ \mathcal{N}) \tag{4.1}$$

$$\Sigma = \{\Phi(x^{\alpha}) = k|k\epsilon\Re\} \tag{4.2}$$

The normal vector field of this family of *hypersurfaces* Σ is defined as :-

Definition 4.1.1.

$$l_{\mu} = -\frac{\partial\Phi}{\partial x^{\mu}} \tag{4.3}$$

$$So,\ l = \tilde{f}g^{\mu\nu}\frac{\partial\Phi}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}\ (for\ given\ \tilde{f}\epsilon\mathcal{F}) \tag{4.4}$$

4.1.1 Category of hypersurfaces based on Signature of their Normal Vector

Let $\mathcal{N}$ be a $n-1$ dimensional hypersurface of a manifold M with normal vector field l. Now, if :-
i) $l^{\mu}l_{\mu} < 0$ at every point $p\epsilon\mathcal{N}$, then $\mathcal{N}$ is said to be a timelike *hypersurface*.
ii) $l^{\mu}l_{\mu} = 0$ at every point $p\epsilon\mathcal{N}$, then $\mathcal{N}$ is said to be a nulllike *hypersurface*.
iii) $l^{\mu}l_{\mu} > 0$ at every point $p\epsilon\mathcal{N}$, then $\mathcal{N}$ is said to be a spacelike *hypersurface*.

4.2 Null Hypersurfaces

Definition 4.2.1. Let $\mathcal{N}$ be a $n-1$ dimensional hypersurface of a manifold M with normal vector field l. If $l^{\mu}l_{\mu} = 0$ at every point $p\epsilon\mathcal{N}$, then $\mathcal{N}$ is said to be a nulllike *hypersurface*.

Note. In a nulllike *hypersurface* the tangent vector fields and the normal vector fields are identical.

Definition 4.2.2. Let $x^{\mu}(\lambda)$ be null geodesics on a nulllike *hypersurface* $\mathcal{N}$ with λ as the affine parameter. Furthermore, let $l^{\mu} = \frac{dx^{\mu}}{d\lambda}$ be tangent to the geodesics which is also normal to $\mathcal{N}$. Then in this case $x^{\mu}(\lambda)$ are called null generators of $\mathcal{N}$.

Lemma 4.2.1. *The null generators of $\mathcal{N}$ are null geodesics.*

Proof. *First let us consider,*

$$\nabla_{\alpha}\partial_{\beta}\Phi = \partial_{\alpha}\partial_{\beta}\Phi - \Gamma^{\mu}_{\alpha\beta}\partial_{\mu}\Phi$$

$$Further, \nabla_{\beta}\partial_{\alpha}\Phi = \partial_{\beta}\partial_{\alpha}\Phi - \Gamma^{\mu}_{\beta\alpha}\partial_{\mu}\Phi$$

But since partial derivatives commute and Γ is symmetric in its lower indices so the above two eq^{n}s are essentially identical. Now consider,

$$\begin{aligned} l^\beta \nabla_\beta l_\alpha &= -\partial^\beta \Phi \nabla_\beta \partial_\alpha \Phi \\ &= -\partial^\beta \Phi \nabla_\alpha \partial_\beta \Phi \\ &= -\frac{1}{2}\nabla_\alpha(\partial^\beta \Phi \partial_\beta \Phi) = -\frac{1}{2}\nabla_\alpha(l^\beta l_\beta) \end{aligned}$$

In the last eq^n since $l^\beta l_\beta$ is zero everywhere on $\mathcal{N}$ it must be proportional to normal vector field. So,

$$\frac{1}{2}\nabla_\alpha(-l^\beta l_\beta) = \kappa l_\alpha$$

$$\nabla_\alpha(-l^\beta l_\beta) = 2\kappa l_\alpha \tag{4.5}$$

This further shows :-

$$l^\beta \nabla_\beta l^\alpha = \kappa l^\alpha \tag{4.6}$$

which is nothing but the geodesic equation. So, since l is tangent to null generators of $\mathcal{N}$ and furthermore, satisfies the geodesic equation; the null generators of $\mathcal{N}$ are indeed the null geodesics. **Proved**.

Now, we present another proof of the given lemma which produces a useful formula for normal vector fields.

Proof. *Consider for some $\tilde{f}\epsilon\mathcal{F}$,*

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \\ Now,\ l^\mu \nabla_\mu l^\nu &= l^\mu \nabla_\mu(\tilde{f} g^{\alpha\nu}\partial_\alpha \Phi) \\ &= g^{\alpha\nu}\partial_\alpha \Phi l^\mu \nabla_\mu \tilde{f} + l^\mu \tilde{f}\partial_\alpha \Phi \nabla_\mu g^{\alpha\nu} + l^\mu \tilde{f} g^{\alpha\nu}\nabla_\mu \partial_\alpha \Phi \\ &= l^\mu g^{\alpha\nu}\partial_\alpha \Phi \partial_\mu \tilde{f} + l^\mu \tilde{f} g^{\alpha\nu}\nabla_\alpha \partial_\mu \Phi \\ &= g^{\alpha\nu}\partial_\alpha \Phi(l^\mu \partial_\mu \tilde{f}) + (l^\mu \nabla_\alpha \partial_\mu \Phi)\tilde{f} g^{\alpha\nu} \\ &= \tilde{f}^{-1} l^\nu (l^\mu \partial_\mu \tilde{f}) + \tilde{f} l^\mu \nabla^\nu \partial_\mu \Phi \\ &= \tilde{f}^{-1} l^\nu (l^\mu \partial_\mu \tilde{f}) + \tilde{f} l^\mu (\nabla^\nu (\tilde{f}^{-1}) l_\mu) \\ &= l^\nu l^\mu \partial_\mu (ln\tilde{f}) + l^\mu \nabla^\nu l_\mu - \tilde{f}^{-1}\partial^\nu \tilde{f} l^2 \\ &= (l\cdot\partial ln\tilde{f}) l^\nu + \frac{1}{2}\nabla^\nu(l^\mu l_\mu) - \tilde{f}^{-1}\partial^\nu \tilde{f} l^2 \\ Now,\ on\ \mathcal{N}\ ;\ &= (l\cdot\partial ln\tilde{f}) l^\nu - 2\kappa l^\nu \end{aligned}$$

So, we get :-

$$l^\mu \nabla_\mu l^\nu\ \alpha\ l^\nu \tag{4.7}$$

which is nothing but the geodesic equation. **Proved**.

Remark. Now, in case of affine parametrization of null geodesics we observe :-

$$\begin{aligned} 2\kappa l^\nu &= (l\cdot\partial ln\tilde{f}) l^\nu \\ \Rightarrow \kappa &= \frac{1}{2}(l\cdot\partial ln\tilde{f}) = l\cdot\partial lnf\ (absorbing\ \frac{1}{2}\ in\ f\epsilon\mathcal{F}) \end{aligned}$$

So, for some $f\epsilon\mathcal{F}$ we obtain the useful relation as :-

$$\kappa = l\cdot\partial lnf \tag{4.8}$$

4.2.1 Killing Horizons

Definition 4.2.3. A null *hypersurface* $\mathcal{N}$ is called a *killing horizon* of *killing vector field* ξ if ξ is a normal vector field of $\mathcal{N}$.

Then, by definition; $\xi\ \alpha\ l\ \Rightarrow\ \xi = fl$ for some $f\epsilon\mathcal{F}$. So, in this case :-

$$\nabla_\alpha(-\xi^\beta \xi_\beta) = 2\kappa_\xi \xi_\alpha \tag{4.9}$$

$$\xi^\beta \nabla_\beta \xi^\alpha = \kappa_\xi \xi^\alpha \tag{4.10}$$

$$\kappa_\xi = \xi\cdot\partial lnf \tag{4.11}$$

This proportionality constant κ_ξ appearing over here is known as the *surface gravity* of the given *killing horizon.*

Note. From here on we would denote the *surface gravity* of the *killing horizons* just by κ.

Alternative definition of surface gravity

Let $\mathcal{N}$ be a null hypersurface and a *killing horizon* of the *killing vector field* ξ of the given geometry. Then, ξ is normal to $\mathcal{N}$. Now, by *Frobenius' theorem* (to be proved later) :-

$$\xi_{[\mu}\nabla_\nu\xi_{\rho]} = 0 \tag{4.12}$$

So,

$$\frac{1}{3!}[\xi_\mu\nabla_\nu\xi_\rho + \xi_\rho\nabla_\mu\xi_\nu + \xi_\nu\nabla_\rho\xi_\mu - \xi_\rho\nabla_\nu\xi_\mu - \xi_\nu\nabla_\mu\xi_\rho - \xi_\mu\nabla_\rho\xi_\nu] = 0$$

$$\begin{aligned}
Killing's\ Equation,\ & \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = 0 \\
&\Rightarrow \frac{2}{3!}[\xi_\mu\nabla_\nu\xi_\rho + \xi_\rho\nabla_\mu\xi_\nu + \xi_\nu\nabla_\rho\xi_\mu] = 0 \\
&\Rightarrow \xi_\mu\nabla_\nu\xi_\rho + \xi_\rho\nabla_\mu\xi_\nu + \xi_\nu\nabla_\rho\xi_\mu = 0 \\
&\Rightarrow \xi_\rho\nabla_\mu\xi_\nu + \xi_\mu\nabla_\nu\xi_\rho - \xi_\nu\nabla_\mu\xi_\rho = 0
\end{aligned}$$

Multiply by $\nabla^\mu\xi^\nu$:-

$$\xi_\rho(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu) + (\nabla^\mu\xi^\nu)\xi_\mu(\nabla_\nu\xi_\rho) - (\nabla^\mu\xi^\nu)\xi_\nu(\nabla_\mu\xi_\rho) = 0$$

So,

$$\begin{aligned}
\xi_\rho(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu) &= -(\nabla^\mu\xi^\nu)\xi_\mu(\nabla_\nu\xi_\rho) - (\nabla^\nu\xi^\mu)\xi_\nu(\nabla_\mu\xi_\rho) \\
&= -2(\nabla^\mu\xi^\nu)\xi_\mu(\nabla_\nu\xi_\rho) \\
&= -2(\xi_\mu\nabla^\mu\xi^\nu)(\nabla_\nu\xi_\rho) \\
&= -2\kappa\xi^\nu\nabla_\nu\xi_\rho \\
&= -2\kappa^2\xi_\rho
\end{aligned}$$

Finally, we get :-

$$\kappa^2 = -\frac{1}{2}\left|(\nabla^\mu\xi^\nu)(\nabla_\mu\xi_\nu)\right|_{\mathcal{N}} \tag{4.13}$$

Now, we consider some examples where we calculate *killing vector fields* and *killing horizons* of some geometries.

Example 4.2.1. Consider $\Re^1 \otimes \Re^1$:-

$$ds^2 = -dt^2 + dx^2$$

From *Killing's equation* we get :-

$$\begin{aligned}
\frac{\partial\xi_x}{\partial x} &= 0 \\
\Rightarrow\ \xi_x &= f(t) \\
And,\ \frac{\partial\xi_t}{\partial t} &= 0 \\
\Rightarrow\ \xi_t &= g(x) \\
Also,\ \frac{\partial\xi_x}{\partial t} + \frac{\partial\xi_t}{\partial x} &= 0 \\
\Rightarrow\ f^{'}(t) + g^{'}(x) &= 0 \\
Possible,\ only\ if;\ f^{'}(t) = -g^{'}(x) &= a(constant)
\end{aligned}$$

So, this gives :-

$$\begin{aligned}
f(t) &= at + c_1 \\
g(x) &= -ax + c_2
\end{aligned}$$

Choosing different values of a, c_1, c_2 gives different linearly independent $sol^n s$ for the *killing vector fields.* Linearly independent *killing vector fields* are :-

$$\begin{aligned}
\xi_1 &= \frac{\partial}{\partial t}\ (a = 0, c_1 = 0, c_2 = 1) \\
\xi_2 &= \frac{\partial}{\partial x}\ (a = 0, c_1 = 1, c_2 = 0) \\
\xi_3 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\ (a = 1, c_1 = 0, c_2 = 0)
\end{aligned}$$

Now, let us consider the family of *hypersurfaces* containing horizons as; $t^2 - x^2 = constant$. Also, we claim that; $\mathcal{N} \equiv t^2 - x^2 = 0$ describes horizons. Let us compute its normal vector fields. For some $\tilde{f} \epsilon \mathcal{F}$ consider;

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu} \frac{\partial(t^2 - x^2)}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\ &= \tilde{f} g^{tt} 2t \frac{\partial}{\partial t} - \tilde{f} g^{xx} 2x \frac{\partial}{\partial x} \\ &= -\tilde{f} 2t \frac{\partial}{\partial t} - \tilde{f} 2x \frac{\partial}{\partial x} \\ Now, |l^\mu l_\mu|_{\mathcal{N}} &= -4\tilde{f}^2 t^2 + 4\tilde{f}^2 x^2 = -4\tilde{f}^2(t^2 - x^2) = 0 \end{aligned}$$

Hence, $t^2 - x^2 = 0$ indeed is a null *hypersurface*. Now, consider;

$$\begin{aligned} \xi &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} \\ |\xi^\mu \xi_\mu|_{\mathcal{N}} &= t^2 - x^2 = 0 \end{aligned}$$

So, $t^2 - x^2 = 0$ gives null *hypersurfaces* which are *killing horizons* to ξ. Now,

$$\begin{aligned} \nabla_\alpha(-\xi^\mu \xi_\mu) &= \partial_\alpha(-\xi^\mu \xi_\mu) \\ &= \partial_\alpha(x^2 - t^2) = 2x\partial_\alpha x - 2t\partial_\alpha t \\ Now,\ \xi_\alpha &= c_1 \partial_\alpha(t^2 - x^2) = c_1(2t\partial_\alpha t - 2x\partial_\alpha x) \\ \xi_t &= 2c_1 t\ \xi_x = -2c_1 x \\ So,\ \xi^\mu \xi_\mu &= -4c_1^2(t^2 - x^2) = t^2 - x^2 \end{aligned}$$

Choosing the +ve sign we get :-

$$\begin{aligned} c_1 &= -\frac{1}{2} \\ So,\ \xi_\alpha &= -\frac{1}{2}(2t\partial_\alpha t - 2x\partial_\alpha x) \end{aligned}$$

Now using $\nabla_\alpha(-\xi^\mu \xi_\mu) = 2\kappa \xi_\alpha$ we get :-

$$2x\partial_\alpha x - 2t\partial_\alpha t = -2\kappa\frac{1}{2}(2t\partial_\alpha t - 2x\partial_\alpha x)$$

So, we finally obtain :-

$$\kappa = 1$$

Example 4.2.2. Consider $\Re^1 \otimes \Re^2$:-

$$ds^2 = -dt^2 + dx^2 + dy^2$$

Form *Killing's equation* we get :-

$$\begin{aligned} \frac{\partial \xi_t}{\partial t} &= 0 \ \Rightarrow\ \xi_t = f(x, y) \\ \frac{\partial \xi_x}{\partial x} &= 0 \ \Rightarrow\ \xi_x = g(y, t) \\ \frac{\partial \xi_y}{\partial y} &= 0 \ \Rightarrow\ \xi_y = h(x, t) \\ Furthermore,\ \frac{\partial \xi_t}{\partial x} + \frac{\partial \xi_x}{\partial t} &= 0 \\ \frac{\partial \xi_x}{\partial y} + \frac{\partial \xi_y}{\partial x} &= 0 \\ \frac{\partial \xi_t}{\partial y} + \frac{\partial \xi_y}{\partial t} &= 0 \\ \Rightarrow\ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial t} &= 0 \\ And,\ \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x} &= 0 \\ Also,\ \frac{\partial f}{\partial y} + \frac{\partial h}{\partial t} &= 0 \end{aligned}$$

So, $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial t}$ are both functions of y. So, let :-

$$\frac{\partial f}{\partial x} = f_1(y); \ \frac{\partial g}{\partial t} = -f_1(y)$$
$$Similarly, \ \frac{\partial g}{\partial y} = g_1(t); \ \frac{\partial h}{\partial x} = -g_1(t)$$
$$And, \ \frac{\partial f}{\partial y} = h_1(x); \ \frac{\partial h}{\partial t} = -h_1(x)$$

So from the above three $eq^n s$ we get three pairs of equations as :-

$$f = f_1(y)x + f_2(y)$$
$$g = -f_1(y)t + f_3(y)$$
$$g = g_1(t)y + g_2(t)$$
$$h = -g_1(t)x + g_3(t)$$
$$f = h_1(x)y + f_4(x)$$
$$h = -h_1(x)t + h_2(x)$$

Comparing 2 f's we get :-

$$f = f_1(y)x + f_2(y) = h_1(x)y + f_4(x)$$
$$\Rightarrow f_2(y) - f_4(x) = h_1(x)y - f_1(y)x$$
$$Possible, \ only \ if; \ h_1(x) = c_5; \ f_1(y) = c_6$$
$$So, \ f_2(y) = yc_5 + c_1; \ f_4(x) = xc_6 + c_1$$

Similarly comparing 2 g's and 2 h's we get :-

$$g_2(t) = -c_6 t + c_2$$
$$f_3(y) = c_3 y + c_2$$
$$g_3(t) = c_4 + tc_5$$
$$h_2(x) = c_4 + xc_3$$

So,

$$f(x, y) = c_6 x + c_5 y + c_1$$
$$g(y, t) = -c_6 t + c_3 y + c_2$$
$$h(x, t) = -c_5 t + c_4 - c_3 x$$

This gives :-

$$\xi = -f\frac{\partial}{\partial t} + g\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}$$
$$= -(c_6 x + c_5 y + c_1)\frac{\partial}{\partial t} + (-c_6 t + c_3 y + c_2)\frac{\partial}{\partial x} + (-c_5 t + c_4 - c_3 x)\frac{\partial}{\partial y}$$

Now choosing different values for the constants we get different linearly independent *killing vector fields* as :-

$$\xi_1 = \frac{\partial}{\partial t} \ (c_1 = 1, c_2 = c_3 = c_4 = c_5 = c_6 = 0)$$
$$\xi_2 = \frac{\partial}{\partial x} \ (c_2 = 1, c_1 = c_3 = c_4 = c_5 = c_6 = 0)$$
$$\xi_3 = \frac{\partial}{\partial y} \ (c_4 = 1, c_2 = c_3 = c_1 = c_5 = c_6 = 0)$$
$$\xi_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \ (c_3 = 1, c_2 = c_1 = c_4 = c_5 = c_6 = 0)$$
$$\xi_5 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} \ (c_5 = -1, c_2 = c_3 = c_4 = c_1 = c_6 = 0)$$
$$\xi_6 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \ (c_6 = -1, c_2 = c_3 = c_4 = c_5 = c_1 = 0)$$

Now, let us consider the family of *hypersurfaces* containing horizons as; $t^2 - x^2 - y^2 = constant$. Also, we claim that; $\mathcal{N} \equiv t^2 - x^2 - y^2 = 0$ describes horizons. Let us compute its normal vector fields. For some $\tilde{f} \epsilon \mathcal{F}$ consider;

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu} \frac{\partial(t^2 - x^2 - y^2)}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\ &= \tilde{f} g^{tt} 2t \frac{\partial}{\partial t} - \tilde{f} g^{xx} 2x \frac{\partial}{\partial x} - \tilde{f} g^{yy} 2y \frac{\partial}{\partial y} \\ &= -\tilde{f} 2t \frac{\partial}{\partial t} - \tilde{f} 2x \frac{\partial}{\partial x} - \tilde{f} 2y \frac{\partial}{\partial y} \\ Now, |l^\mu l_\mu|_{\mathcal{N}} &= -4\tilde{f}^2 t^2 + 4\tilde{f}^2 x^2 + 4\tilde{f}^2 y^2 = -4\tilde{f}^2 (t^2 - x^2 - y^2) = 0 \end{aligned}$$

Hence, $t^2 - x^2 - y^2 = 0$ indeed is a null *hypersurface*.

Example 4.2.3. Consider 2−sphere metric :-

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

Now let us compute non-vanishing components of the *Affine connection*.

$$\begin{aligned} \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^\phi_{\theta\phi} &= \Gamma^\phi_{\phi\theta} = \cot\theta \end{aligned}$$

Now from *Killing's equation* we get :-

$$\begin{aligned} &\frac{\partial\xi_\theta}{\partial\theta} = 0 \;\Rightarrow\; \xi_\theta = f(\theta) \\ And, \; &\frac{\partial\xi_\phi}{\partial\phi} = -\sin\theta\cos\theta\xi_\theta \\ &= -\sin\theta\cos\theta f(\phi) \;\Rightarrow\; \xi_\phi = -\sin\theta\cos\theta g(\phi) + h(\theta) \; (where, \; g^{'}(\phi) = f(\phi)) \\ Also, \; &\frac{\partial\xi_\theta}{\partial\phi} + \frac{\partial\xi_\phi}{\partial\theta} = 2\cot\theta\xi_\phi \\ &\Rightarrow f^{'}(\phi) + g(\phi)[\sin^2\phi - \cos^2\phi] + h^{'}(\theta) = -2\cos^2\theta g(\phi) + 2\cot\theta h(\theta) \end{aligned}$$

Now take $h = 0$, so from above equation;

$$\begin{aligned} f^{'}(\phi) + g(\phi) &= 0 \\ Then, \; f^{''}(\phi) &= -f(\phi) \\ So, \; f(\phi) &= a\sin\phi + b\cos\phi \\ And, \; g(\phi) &= b\sin\phi - a\cos\phi \end{aligned}$$

This gives :-

$$\begin{aligned} \xi_\theta &= a\sin\phi + b\cos\phi \\ \xi_\phi &= \sin\theta\cos\theta(a\cos\phi - b\sin\phi) \\ \xi^\theta &= a\sin\phi + b\cos\phi \\ \xi^\phi &= \cot\theta(a\cos\phi - b\sin\phi) \end{aligned}$$

where $a, b\epsilon\Re$.

So,

$$\begin{aligned} \xi &= \xi^\theta \frac{\partial}{\partial\theta} + \xi^\phi \frac{\partial}{\partial\phi} \\ &= (a\sin\phi + b\cos\phi)\frac{\partial}{\partial\theta} + \cot\theta(a\cos\phi - b\sin\phi)\frac{\partial}{\partial\phi} \end{aligned}$$

Choosing different values for a and b gives different linearly dependent *killing vector fields* as :-

$$\begin{aligned} \xi_1 &= \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \\ \xi_2 &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \end{aligned}$$

Furthermore, $\xi_\theta = 0$ also yields $\frac{\partial \xi_\phi}{\partial \phi} = 0$. This gives :-

$$\begin{aligned} \xi_\phi &= f_1(\theta) \\ So,\ \frac{\partial f_1}{\partial \theta} &= 2\cot\theta f_1 \\ \Rightarrow\ f_1(\theta) &= \sin^2\theta \\ So,\ \xi^\phi &= g^{\phi\phi}\xi_\phi = 1 \end{aligned}$$

Hence, we finally obtain :-

$$\xi_3 = \frac{\partial}{\partial \phi}$$

Chapter 5

Congruences of Geodesics

Definition 5.0.4. Congruence of geodesics is a family of geodesics on a manifold M such that through every point $p\epsilon M$ there passes only one geodesic. So, they are non-intersecting family of geodesics. In general, congruences can be defined for any curve.

5.1 Stress Tensor Properties

Stress tensor $T^{\alpha\beta}$ as we know being a type $(2,0)$ tensor can be decomposed as :-

$$T^{\alpha\beta} = \rho\hat{e}_0^\alpha\hat{e}_0^\beta + p_1\hat{e}_1^\alpha\hat{e}_1^\beta + p_2\hat{e}_2^\alpha\hat{e}_2^\beta + p_3\hat{e}_3^\alpha\hat{e}_3^\beta \tag{5.1}$$

where, $\rho - energy - matter\ density$ and $p_1, p_2, p_3 - principal\ pressures$. Also, $\{\hat{e}_\mu^\alpha = \frac{\partial x^\alpha}{\partial \chi^\mu}\}-$orthonormal basis (usual dual basis vectors) and χ^μ denotes *locally inertial frame*. So, now;

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}g_{\mu\nu} \tag{5.2}$$

$$In\ locally\ inertial\ frame,\ \eta_{\mu\nu} = \frac{\partial x^\alpha}{\partial \chi^\mu}\frac{\partial x^\beta}{\partial \chi^\nu}g_{\alpha\beta} \tag{5.3}$$

$$So,\ \eta_{\mu\nu} = \hat{e}_\mu^\alpha\hat{e}_\nu^\beta g_{\alpha\beta} \tag{5.4}$$

$$For\ the\ inverse,\ g^{\alpha\beta} = \hat{e}_\mu^\alpha\hat{e}_\nu^\beta\eta^{\mu\nu} \tag{5.5}$$

For a perfect fluid, $p_1 = p_2 = p_3 \equiv p$. Substituting this in Eq^n 5.1 and using 5.5 we get :-

$$\begin{aligned} T^{\alpha\beta} &= \rho\hat{e}_0^\alpha\hat{e}_0^\beta + p(\hat{e}_1^\alpha\hat{e}_1^\beta + \hat{e}_2^\alpha\hat{e}_2^\beta + \hat{e}_3^\alpha\hat{e}_3^\beta) \\ &= \rho\hat{e}_0^\alpha\hat{e}_0^\beta + p(g^{\alpha\beta} + \hat{e}_0^\alpha\hat{e}_0^\beta) \\ &= (\rho + p)\hat{e}_0^\alpha\hat{e}_0^\beta + pg^{\alpha\beta} \end{aligned}$$

where $\hat{e}_0^\alpha$ can be identified as the $4-$velocity of the perfect fluid.

Now, for an arbitrary normalized timelike vector :-

$$u^\alpha = \gamma(\hat{e}_0^\alpha + a\hat{e}_1^\alpha + b\hat{e}_2^\alpha + c\hat{e}_3^\alpha) \tag{5.6}$$

where $\gamma = (1 - a^2 - b^2 - c^2)^{-\frac{1}{2}}$ along with the constraint $(a^2 + b^2 + c^2 < 1)$.

Now, for an arbitrary nulllike vector (whose normalization is clearly arbitrary) :-

$$k^\alpha = \hat{e}_0^\alpha + a'\hat{e}_1^\alpha + b'\hat{e}_2^\alpha + c'\hat{e}_3^\alpha \tag{5.7}$$

with the constraint $(a'^2 + b'^2 + c'^2 = 1)$.

5.1.1 Energy Conditions

Weak Energy Condition

For any energy-matter distribution given an observer with $4-$velocity as u^α; the energy-matter density ρ is evidently given by :-

$$\rho = T_{\alpha\beta}u^\alpha u^\beta \tag{5.8}$$

The weak energy condition (positivity of energy-matter density) is given by :-

$$T_{\alpha\beta}u^{\alpha}u^{\beta} \geq 0 \tag{5.9}$$

This gives :-

$$\gamma^2(\rho\hat{e}^0_\alpha\hat{e}^0_\beta + p_1\hat{e}^1_\alpha\hat{e}^1_\beta + p_2\hat{e}^2_\alpha\hat{e}^2_\beta + p_3\hat{e}^3_\alpha\hat{e}^3_\beta)(\hat{e}^\alpha_0 + a\hat{e}^\alpha_1 + b\hat{e}^\alpha_2 + c\hat{e}^\alpha_3)(\hat{e}^\beta_0 + a\hat{e}^\beta_1 + b\hat{e}^\beta_2 + c\hat{e}^\beta_3) \geq 0$$
$$\Rightarrow \gamma^2(\rho + a^2p_1 + b^2p_2 + c^2p_3) \geq 0$$

where $a^2 + b^2 + c^2 < 1$. Now, let us choose, $a = b = c = 0$, then, $\rho \geq 0$. Now, $b = c = 0$ gives $\rho + a^2p_1 \geq 0$; but, $a^2 < 1$. This implies, $0 \leq \rho + a^2p_1 \leq \rho + p_1$. So, $\rho + p_1 > 0$. Similarly choosing b and c alternatively, we get; $\rho + p_i > 0 \;\forall\; i\epsilon\{1, 2, 3\}$.

Hence, weak energy condition gives :-

$$T_{\alpha\beta}u^{\alpha}u^{\beta} \geq 0$$
$$\Rightarrow \rho \geq 0 \tag{5.10}$$
$$Furthermore,\; \rho + p_i > 0 \tag{5.11}$$

Null Energy Condition

For arbitrary null vector k and any energy-matter distribution, the null energy condition reads :-

$$T_{\alpha\beta}k^{\alpha}k^{\beta} = 0 \tag{5.12}$$

This gives similarly as above :-

$$\rho + a^{'2}p_1 + b^{'2}p_2 + c^{'2}p_3 \geq 0$$

where $a^2 + b^2 + c^2 < 1$. Now, let $b^{'} = c^{'} = 0$; then, $a^{'2} = 1$. So, $\rho + p_1 \geq 0$. Similarly choosing b and c alternatively, we get; $\rho + p_i \geq 0 \;\forall\; i\epsilon\{1, 2, 3\}$.

Hence, weak energy condition gives :-

$$T_{\alpha\beta}k^{\alpha}k^{\beta} \geq 0$$
$$\Rightarrow \rho + p_i \geq 0 \tag{5.13}$$

Strong Energy Condition

For any energy-matter distribution given an observer with 4−velocity as u^μ; the strong energy condition reads :-

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)u^{\mu}u^{\nu} \geq 0 \tag{5.14}$$

where $T = g_{\mu\nu}T^{\mu\nu}$. Now, let us consider *Einstein's Field Equations* :-

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$
$$Contracting\ with\ g^{\mu\nu},\; R - 2R = 8\pi T$$
$$\Rightarrow R = -8\pi T$$
$$So,\; R_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}(8\pi T) + 8\pi T_{\mu\nu} = -4\pi g_{\mu\nu}T + 8\pi T_{\mu\nu}$$
$$\Rightarrow R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

So,

$$T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T = \frac{R_{\mu\nu}}{8\pi} \tag{5.15}$$

So, the strong energy condition becomes :-

$$R_{\mu\nu}u^\mu u^\nu \geq 0 \tag{5.16}$$

Now,

$$\begin{aligned}
g_{\mu\nu}u^\mu u^\nu &= \gamma^2 g_{\mu\nu}(\hat{e}_0^\mu + a\hat{e}_1^\mu + b\hat{e}_2^\mu + c\hat{e}_3^\mu)(\hat{e}_0^\nu + a\hat{e}_1^\nu + b\hat{e}_2^\nu + c\hat{e}_3^\nu) \\
&= \gamma^2(g_{\mu\nu}\hat{e}_0^\mu\hat{e}_0^\nu + a^2 g_{\mu\nu}\hat{e}_1^\mu\hat{e}_1^\nu + b^2 g_{\mu\nu}\hat{e}_2^\mu\hat{e}_2^\nu + c^2 g_{\mu\nu}\hat{e}_3^\mu\hat{e}_3^\nu) \\
&= -\gamma^2(1 - a^2 - b^2 - c^2) = -1
\end{aligned}$$

So, from strong energy condition we get :-

$$T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T \tag{5.17}$$

Now,

$$\begin{aligned}
T &= g_{\mu\nu}T^{\mu\nu} \\
&= g_{\mu\nu}(\rho\hat{e}_0^\mu\hat{e}_0^\nu + p_1\hat{e}_1^\mu\hat{e}_1^\nu + p_2\hat{e}_2^\mu\hat{e}_2^\nu + p_3\hat{e}_3^\mu\hat{e}_3^\nu) \\
&= -\rho + p_1 + p_2 + p_3
\end{aligned}$$

So, we get :-

$$-\frac{1}{2}T = \frac{1}{2}(\rho - p_1 - p_2 - p_3)$$

$$So,\ by\ strong\ energy\ condition;\ \gamma^2(\rho + a^2p_1 + b^2p_2 + c^2p_3) \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3) \tag{5.18}$$

Furthermore, $a = b = c = 0$ gives $\gamma = 1$ and;

$$\begin{aligned}
&\rho \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3) \\
&\rho + p_1 + p_2 + p_3 \geq 0 \\
&\rho + \sum_{i=1}^{3} p_i \geq 0
\end{aligned}$$

Also, $b = c = 0$ gives :-

$$\begin{aligned}
&\frac{1}{1-a^2}(\rho + a^2p_1) \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3) \\
&\rho + a_1^p \geq \frac{1}{2}(\rho - p_1 - p_2 - p_3)(1 - a^2) \geq \\
\Rightarrow\ &\rho + p_1 + p_2 + p_3 \geq a^2(p_2 + p_3 - p_1 - \rho) \\
\Rightarrow\ &\rho(1 + a^2) + p_1(1 + a^2) \geq (a^2 - 1)p_2 + (a^2 - 1)p_3
\end{aligned}$$

Since $a^2 < 1$, $\rho + p_1 \geq 0$. Similarly choosing b and c alternatively, we get; $\rho + p_i \geq 0\ \forall\ i\epsilon\{1, 2, 3\}$.

Hence, strong energy condition gives :-

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)u^\mu u^\nu \geq 0$$

$$\Rightarrow \rho + \sum_{i=1}^{3} p_i \geq 0 \tag{5.19}$$

$$Also,\ \rho + p_i \geq 0 \tag{5.20}$$

Dominant Energy Condition

We know momentum density of energy-matter distribution as observed by observer with $4-$velocity u^α is $-T^\alpha_\beta u^\beta$. Now, the dominant energy condition reads :-

"$-T^{\alpha}_{\beta}u^{\beta}$ must be nulllike or timelike, i.e.; energy-matter distribution must be along nulllike or timelike world lines. Furthermore, $-T^{\alpha}_{\beta}u^{\beta}$ must be future directed too."

Now,

$$\begin{aligned} -T^{\alpha}_{\beta} &= -\gamma(\rho\hat{e}^{\alpha}_{0}\hat{e}^{0}_{\beta} + p_1\hat{e}^{\alpha}_{1}\hat{e}^{1}_{\beta} + p_2\hat{e}^{\alpha}_{2}\hat{e}^{2}_{\beta} + p_3\hat{e}^{\alpha}_{3}\hat{e}^{3}_{\beta})(\hat{e}^{\beta}_{0} + a\hat{e}^{\beta}_{1} + b\hat{e}^{\beta}_{2} + c\hat{e}^{\beta}_{3}) \\ &= -(\rho\hat{e}^{\alpha}_{0} + ap_1\hat{e}^{\alpha}_{1} + bp_2\hat{e}^{\alpha}_{2} + cp_3\hat{e}^{\alpha}_{3}) \end{aligned}$$

Now let us calculate the norm of $-T^{\alpha}_{\beta}u^{\beta}$ and constraint it to be nulllike or timelike :-

$$\begin{aligned} &g_{\alpha\beta}(\rho\hat{e}^{\alpha}_{0} + ap_1\hat{e}^{\alpha}_{1} + bp_2\hat{e}^{\alpha}_{2} + cp_3\hat{e}^{\alpha}_{3})(\rho\hat{e}^{\beta}_{0} + ap_1\hat{e}^{\beta}_{1} + bp_2\hat{e}^{\beta}_{2} + cp_3\hat{e}^{\beta}_{3}) = -\rho^2 + a^2p_1^2 + b^2p_2^2 + c^2p_3^2 \leq 0 \\ \Rightarrow\ &\rho^2 - a^2p_1^2 - b^2p_2^2 - c^2p_3^2 \geq 0 \end{aligned}$$

Now, $a = b = c = 0$ gives $\rho^2 \geq 0$ and future directed further gives $\rho \geq 0$. Furthermore, $b = c = 0$ gives :-

$$\begin{aligned} \rho^2 - a^2p_1^2 &\geq 0 \\ Since,\ a^2 < 1;\ \rho &\geq |ap_1| \\ So,\ \rho &\geq |p_1| \end{aligned}$$

Similarly choosing b and c alternatively, we get; $\rho \geq p_i\ \forall\ i\epsilon\{1, 2, 3\}$.

Hence, dominant energy condition gives :-

$-T^{\alpha}_{\beta}u^{\beta}$ is timelike or nulllike and future-directed.

$$\Rightarrow \rho \geq 0 \tag{5.21}$$

$$Also,\ \rho \geq |p_i| \tag{5.22}$$

5.2 Deformable Medium

Consider the following figure :-

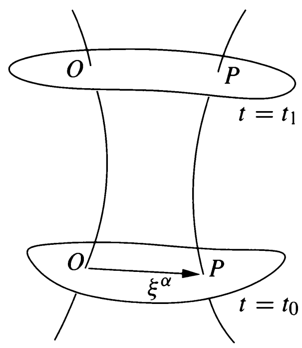

Figure 5.1: Two dimensional deformable medium

Let the reference point be O. Consider a small time dependent displacement around it as $\xi^{\alpha}(t)$. Now,

$$\frac{d\xi^{\alpha}(t)}{dt} = B^{\alpha}_{\beta}\xi^{\beta}(t) \tag{5.23}$$

for some tensor B^{α}_{β}. Furthermore,

$$\xi^{\beta}(t_1) = \xi^{\beta}(t_0) + \Delta\xi^{\beta}(t_0) \tag{5.24}$$

$$\Delta\xi^{\alpha}(t_0) = B^{\alpha}_{\beta}\xi^{\beta}(t_0)\Delta t \tag{5.25}$$

where $\Delta t = t_1 - t_0$.

Now, for simplicity let us consider; $\xi^{\alpha}(t_0) = r_0(\cos\phi, \sin\phi)$ where $r_0\epsilon\Re$ and $\phi\epsilon[0, 2\pi)$.

5.2.1 Expansion Scalar

Let,

$$B^{\alpha}_{\beta} = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} \ \alpha \ I_2$$

where $\theta \epsilon \Re$ and I_2 be 2x2 Identity matrix. So, $B^{\alpha}_{\alpha} = \theta$.

Now,

$$\begin{aligned} \Delta\xi^{\alpha}(t_0) &= \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} r_0 \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \Delta t \\ &= \frac{r_0\theta}{2}\Delta t\theta(\cos\phi, \sin\phi) \end{aligned}$$

So,

$$\begin{aligned} \xi^{\alpha}(t_1) &= r_0(\cos\phi, \sin\phi) + \frac{r_0\theta}{2}\Delta t\theta(\cos\phi, \sin_{\phi}) \\ &= r_0\left(1 + \frac{\theta}{2}\Delta t\right)(\cos\phi, \sin\phi) \end{aligned}$$

Let us now calculate the initial and final areas :-

$$\begin{aligned} A_i &= \pi r_0^2 \\ A_f &= \pi(r_0\left(1 + \frac{\theta}{2}\Delta t\right))^2 \\ \Delta A &= \pi\theta r_0^2\Delta t \end{aligned}$$

So, this gives :-

$$\theta = \frac{1}{\pi r_0^2}\frac{\Delta A}{\Delta t} = \frac{1}{A_i}\frac{dA}{dt}$$

5.2.2 Shear Tensor

Let,

$$B^{\alpha}_{\beta} = \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix}$$

where $\sigma_+, \sigma_x \epsilon \Re$ and B^{α}_{β} is tracefree.

Now,

$$\begin{aligned} \Delta\xi^{\alpha}(t_0) &= B^{\alpha}_{\beta}\xi^{\beta}(t_0)\Delta t \\ &= \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix} r_0 \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \Delta t \\ &= r_0\Delta t \begin{pmatrix} \sigma_+\cos\phi + \sigma_x\sin\phi \\ \sigma_x\cos\phi - \sigma_+\sin\phi \end{pmatrix} \end{aligned}$$

So,

$$\begin{aligned} \xi^{\alpha}(t_1) &= r_0 \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + r_0\Delta t \begin{pmatrix} \sigma_+\cos\phi + \sigma_x\sin\phi \\ \sigma_x\cos\phi - \sigma_+\sin\phi \end{pmatrix} \\ &= \begin{pmatrix} r_0(1 + \sigma_+\Delta t)\cos\phi + r_0\Delta t\sigma_x\sin\phi \\ r_0(1 - \sigma_+\Delta t)\sin\phi + r_0\Delta t\sigma_x\cos\phi \end{pmatrix} \end{aligned}$$

Now,

$$
\begin{aligned}
x &= r_0\cos\phi + r_0\Delta t\sigma_+\cos\phi + r_0\Delta t\sigma_x\sin\phi \\
x &= r_0\sin\phi - r_0\Delta t\sigma_+\sin\phi + r_0\Delta t\sigma_x\cos\phi \\
x^2+y^2 &= r_0^2 + 2r_0^2\Delta t\sigma_+\cos 2\phi + 2r_0^2\Delta t\sigma_x\sin 2\phi \\
So,\ r_1(\phi) &= r_0(1+\Delta t\sigma_+\cos 2\phi + \Delta t\sigma_x\sin 2\phi)
\end{aligned}
$$

Let us now calculate the area :-

$$
\begin{aligned}
A &= \int_{r_i}^{r_f}\int_{\phi_i}^{\phi_f} r(\phi)drd\phi \\
&= \int_{r_i}^{r_f}\int_{\phi_i}^{\phi_f} r(1+\Delta t\sigma_+\cos 2\phi + \Delta t\sigma_x\sin 2\phi)drd\phi \\
&= \int_0^{r_0} rdr\int_0^{2\pi} d\phi + \int_0^{r_0} rdr\int_0^{2\pi}\Delta t\sigma_+\cos 2\phi d\phi + \int_0^{r_0} rdr\int_0^{2\pi}\Delta t\sigma_x\sin 2\phi d\phi
\end{aligned}
$$

In the last eq^n the last term goes to zero as they contain integration of $\cos\phi$ and $\sin\phi$ over $\phi\epsilon[0,2\pi)$. So, shear transformation preserves area.

5.2.3 Rotation Tensor

Let,

$$
B^\alpha_\beta = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}
$$

where $\omega\epsilon\Re$ and B^α_β is antisymmetric.

Now,

$$
\begin{aligned}
\Delta\xi^\alpha(t_0) &= \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} r_0 \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}\Delta t \\
&= r_0\omega\Delta t\begin{pmatrix} \sin\phi \\ -\cos\phi \end{pmatrix}
\end{aligned}
$$

Now,

$$
\xi^\alpha(t_1) = \begin{pmatrix} r_0\cos\phi + r_0\omega\Delta t\sin\phi \\ r_0\sin\phi - r_0\omega\Delta t\cos\phi \end{pmatrix}
$$

Now, let us calculate the area :-

$$
\begin{aligned}
x &= r_0\cos\phi + r_0\omega\Delta t\sin\phi \\
y &= r_0\sin\phi - r_0\omega\Delta t\cos\phi \\
x^2+y^2 &= r_0^2 + 2r_0^2\omega\Delta t\cos\phi\sin\phi - 2r_0^2\omega\Delta t\cos\phi\sin\phi = r_0^2
\end{aligned}
$$

So, integrating the last eq^n to get area indeed shows that rotation preserves area.

5.2.4 General form of B^α_β

Now,

$$
B^\alpha_\beta = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix} + \begin{pmatrix} \sigma_+ & \sigma_x \\ \sigma_x & -\sigma_+ \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \tag{5.26}
$$

$$
Now,\ B_{\alpha\beta} = \frac{1}{2}\theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{5.27}
$$

So, this gives :-

$$
\theta = B^\alpha_\alpha \tag{5.28}
$$

Furthermore,

$$\begin{aligned}
B_{(\alpha\beta)} &= \frac{1}{2!}(B_{\alpha\beta} + B_{\beta\alpha}) \\
&= \frac{1}{2}\left(\frac{1}{2}\theta\delta_{\alpha\beta} + \frac{1}{2}\theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \sigma_{\beta\alpha} + \omega_{\alpha\beta} + \omega_{\beta\alpha}\right) \\
&= \frac{1}{2}(\theta\delta_{\alpha\beta} + 2\sigma_{\alpha\beta}) \\
&= \frac{\theta}{2}\delta_{\alpha\beta} + \sigma_{\alpha\beta}
\end{aligned}$$

So,

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{2}\theta\delta_{\alpha\beta}$$

Consider,

$$\begin{aligned}
B_{[\alpha\beta]} &= \frac{1}{2!}(B_{\alpha\beta} - B_{\beta\alpha}) \\
&= \omega_{\alpha\beta}
\end{aligned}$$

So,

$$\omega_{\alpha\beta} = B_{[\alpha\beta]} \tag{5.29}$$

Similarly, in 3−dimensions :-

$$B_{\alpha\beta} = \frac{1}{3}\theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \tag{5.30}$$

$$\Rightarrow \theta = B^{\alpha}_{\alpha} \tag{5.31}$$

$$\Rightarrow \sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{2}\theta\delta_{\alpha\beta} \tag{5.32}$$

$$\Rightarrow \omega_{\alpha\beta} = B_{[\alpha\beta]} \tag{5.33}$$

5.3 Congruence of Timelike Geodesics

Definition 5.3.1. It is a family of non-intersecting timelike geodesics on a manifold M such that their tangent vector fields are timelike.

Let $g_{\alpha\beta}$ be the metric. u^{α} is the tangent vector to the geodesics and ξ^{α} is the deviation vector.

So,

$$u^{\alpha}u_{\alpha} = -1 \tag{5.34}$$

$$u^{\beta}\nabla_{\beta}\xi^{\alpha} = \xi^{\beta}\nabla_{\beta}u^{\alpha} \tag{5.35}$$

$$u^{\beta}\nabla_{\beta}u^{\alpha} = 0 \tag{5.36}$$

$$u^{\alpha}\xi_{\alpha} = 0 \tag{5.37}$$

5.3.1 Transverse Metric

The transverse metric $h_{\alpha\beta}$ is given by :-

$$h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}u_{\beta} \tag{5.38}$$

Consider its orthogonality to the tangent vectors,

$$h_{\alpha\beta}u^{\beta} = g_{\alpha\beta}u^{\beta} + u_{\alpha}u_{\beta}u^{\beta} = u_{\alpha} - u_{\alpha} = 0 \tag{5.39}$$

$$u^{\alpha}h_{\alpha\beta} = u^{\alpha}g_{\alpha\beta} + u^{\alpha}u_{\alpha}u_{\beta} = u_{\beta} - u_{\beta} = 0 \tag{5.40}$$

Now consider *Lorentz frame*;

$$\begin{aligned}
g_{\alpha\beta} &= diag(-1, 1, 1, 1) \\
u_{\alpha} &= diag(-1, 0, 0, 0) \\
So,\ h_{\alpha\beta} &= diag(0, 1, 1, 1)\ (purely\ 3\ dimensional)
\end{aligned}$$

Furthermore, consider :-

$$\begin{aligned} g^{\alpha\mu}h_{\alpha\beta} &= h^{\mu}_{\beta} = \delta^{\mu}_{\beta} + u^{\mu}u_{\beta} \\ So,\ h^{\mu}_{\mu} &= \delta^{\mu}_{\mu} + u^{\mu}u_{\mu} = 4 - 1 = 3 \\ Also,\ h^{\alpha}_{\mu} &= \delta^{\alpha}_{\mu} + u^{\alpha}u_{\mu} \\ Now,\ h^{\alpha}_{\mu}h^{\mu}_{\beta} &= \delta^{\alpha}_{\beta} + 2u^{\alpha}u_{\beta} - u^{\alpha}u_{\beta} = \delta^{\alpha}_{\beta} + u^{\alpha}u_{\beta} = h^{\alpha}_{\beta} \end{aligned}$$

So, we get :-

$$h^{\alpha}_{\alpha} = 3 \tag{5.41}$$

$$h^{\alpha}_{\mu}h^{\mu}_{\beta} = h^{\alpha}_{\beta} \tag{5.42}$$

Hence, $h_{\alpha\beta}$ as defined above is indeed the transverse metric which is transverse to the geodesics and the transverse space is indeed 3−dimensional.

Now define;

$$B_{\alpha\beta} = \nabla_{\beta}u_{\alpha} \tag{5.43}$$

So, $u^{\beta}\nabla_{\beta}u_{\alpha} = u^{\beta}B_{\alpha\beta} = B_{\alpha\beta}u^{\beta}$. So,

$$\begin{aligned} B_{\alpha\beta} &= \frac{1}{3}\theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\ \Rightarrow \theta &= B^{\alpha}_{\alpha} \\ \Rightarrow \sigma_{\alpha\beta} &= B_{(\alpha\beta)} - \frac{1}{2}\theta\delta_{\alpha\beta} \\ \Rightarrow \omega_{\alpha\beta} &= B_{[\alpha\beta]} \end{aligned}$$

For the physical significance of $B_{\alpha\beta}$ consider; $u^{\beta}\nabla_{\beta}\xi_{\alpha} = \xi^{\beta}\nabla_{\beta}u_{\alpha} = \xi^{\beta}B_{\alpha\beta}$. So, it is clear that $B_{\alpha\beta}$ measures the failure of ξ^{α} to be parallely transported along the geodesics.

5.3.2 Frobenius' Theorem for timelike geodesics

Consider timelike geodesics which are *hypersurface* orthogonal (to be defined soon) in the following figure :-

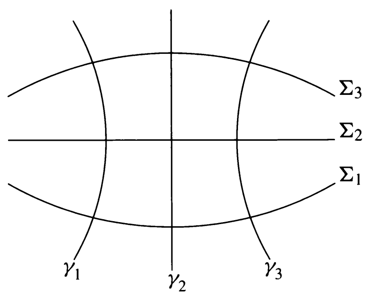

Figure 5.2: γs denote geodesics and Σs denote *hypersurfaces.*

Now, $\Phi(x^{\alpha}) = constant$ describes a family of *hypersurfaces.* $\Phi_{,\alpha}$ is normal to *hypersurface.* Now, we will define what we mean by geodesics being *hypersurface* orthogonal.

Definition 5.3.2. If tangent vector fields of the geodesics are proportional to normal vector fields of the *hypersurfaces* then the geodesics are called *hypersurface* orthogonal geodesics, i.e.;

$$u_{\alpha} = -\mu\Phi_{,\alpha} \tag{5.44}$$

for some $\mu\epsilon\Re$. So, $hypersurface\ orthogonal \Rightarrow u_{\alpha} = -\mu\Phi_{,\alpha}$.

Now consider,

$$u_{[\alpha}\nabla_\beta u_{\gamma]} = \frac{1}{3!}[u_\alpha\nabla_\beta u_\gamma + u_\gamma\nabla_\alpha u_\beta + u_\beta\nabla_\gamma u_\alpha - u_\gamma\nabla_\beta u_\alpha - u_\alpha\nabla_\gamma u_\beta - u_\beta\nabla_\alpha u_\gamma]$$
$$Sunstitute,\ u_\alpha = -\mu\Phi_{,\alpha}\ and\ \nabla_\beta = \partial_\beta$$

Then explicit calculation gives :-

$$u_{[\alpha}\nabla_\beta u_{\gamma]} = 0 \tag{5.45}$$

So, if u_α is normal to a *hypersurface* the above condition holds. Converse, is also true but we won't be proving that.

Now,

$$\begin{aligned}3!(u_{[\alpha}\nabla_\beta u_{\gamma]}) &= 2[u_\alpha B_{[\gamma\beta]} + u_\gamma B_{[\beta\alpha]} + u_\beta B_{[\alpha\gamma]}]\\ &= 2[u_\alpha\omega_{\gamma\beta} + u_\gamma\omega_{\beta\alpha} + u_\beta\omega_{\alpha\gamma}] = 0\end{aligned}$$

So, we get :-

$$u_\alpha\omega_{\gamma\beta} + u_\gamma\omega_{\beta\alpha} + u_\beta\omega_{\alpha\gamma} = 0 \tag{5.46}$$

Now,

$$u^\alpha B_{\alpha\beta} = u^\alpha\nabla_\beta u_\alpha = \frac{1}{2}\nabla_\beta(u^\alpha u_\alpha) = 0 \tag{5.47}$$
$$B_{\alpha\beta}u^\beta = \nabla_\beta u_\alpha u^\beta = u^\beta\nabla_\beta u_\alpha = 0 \tag{5.48}$$

So, $u^\alpha B_{\alpha\beta} = B_{\alpha\beta}u^\beta = 0$ and hence, $B_{\alpha\beta}$ is purely transversal. Since $h_{\alpha\beta}$ is also transversal we see that $\sigma_{\alpha\beta} = \frac{1}{3}\theta h_{\alpha\beta} - B_{(\alpha\beta)}$ is also transversal and so is $\omega_{\alpha\beta}$. Hence, multiplying Eq^n 5.44 by u^γ we get :-

$$u^\gamma u_\alpha\omega_{\gamma\beta} + u^\gamma u_\gamma\omega_{\beta\alpha} + u^\gamma u_\beta\omega_{\alpha\gamma} = 0$$

The first two terms in the last equation vanish as $\omega_{\alpha\beta}$ is transversal and we are left with :-

$$\omega_{\alpha\beta} = 0 \tag{5.49}$$

So, finally we state *Frobenius theorem* for timelike geodesics :-

Theorem 5.3.1. *Let γs be timelike geodesics which are hypersurface orthogonal with tangent vector as u. Furthermore, let ω be the rotation tensor. Then,*

$$hypersurface\ orthogonal\ \Leftrightarrow\ u_{[\alpha}\nabla_\beta u_{\gamma]} = 0\ \Rightarrow\ \omega_{\alpha\beta} = 0 \tag{5.50}$$

We proved the above theorem completely except the converse requirement. We wouldn't be requiring the converse requirement in our discussions.

5.3.3 Interpretation of θ for timelike geodesics

Consider the following figure :-
In the above figure, Σ denotes family of *hypersurfaces* and $\delta\Sigma(\tau_p)$ (where $\tau\epsilon\Re$ is geodesic parameter and $p\epsilon\Sigma$) is a set containing points in small *nbd* of p such that :-
a) Through all points $p^{'}\epsilon\delta\Sigma(\tau_p)$; $\tau = \tau_p$.
b) Through different points $p^{'}\epsilon\delta\Sigma(\tau_p)$ different geodesics pass through, from the given congruence.

Proper time parametrization of geodesics with τ is done in such a way that atleast a γ is orthogonal to $\delta\Sigma(\tau_p)$. Furthermore, $\tau = \tau_p$ is the *hypersurface* and is the congruence cross-section around γ at proper time $\tau = \tau_p$.

Different points on $\delta\Sigma(\tau_p)$ are labelled as $y^a, a\epsilon\{1,2,3\}$. So, each γ can be labelled with y^a so that y^a is

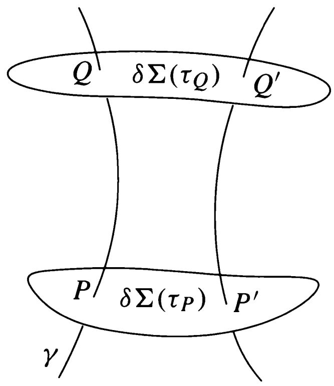

Figure 5.3: Congruence's cross-section about a reference geodesic.

constant along γ and coordinate system thus obtained is (τ, y^a). So, $x^\alpha = x^\alpha(\tau, y^a)$ is the coordinate transformation. Now, define;

$$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \tau}\right)_{y^a} \quad (tangent\ vector\ field) \tag{5.51}$$

$$e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_\tau \quad (transverse\ deviation\ vector) \tag{5.52}$$

Now,

$$u_\alpha e^\alpha_a = 0 \ (by\ parametrization) \tag{5.53}$$

$$u^\mu \nabla_\mu e^\alpha_a = e^\mu_a \nabla_\mu u^\alpha \tag{5.54}$$

$$\begin{aligned} Consider,\ \pounds_u e^\alpha_a &= \frac{\partial e^\alpha_a}{\partial x^\mu} u^\mu - \frac{\partial u^\alpha}{\partial x^\mu} e^\mu_a \\ &= \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\alpha}{\partial y^a}\right)\frac{\partial x^\mu}{\partial \tau} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\alpha}{\partial \tau}\right)\frac{\partial x^\mu}{\partial y^a} \\ &= \frac{\partial^2 x^\alpha}{\partial \tau \partial y^a} - \frac{\partial^2 x^\alpha}{\partial y^a \partial \tau} = 0 \end{aligned} \tag{5.55}$$

Define a 3−tensor (i.e.; scalar under $x^\alpha \to x^{'\alpha}$ but tensor under $y^a \to y^{'a}$) by :-

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \tag{5.56}$$

On γ; $u_\alpha e^\alpha_a = 0$. So using transverse metric $h_{\alpha\beta}$ we get;

$$h_{ab} = h_{\alpha\beta} e^\alpha_a e^\beta_b \tag{5.57}$$

$$h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b \tag{5.58}$$

Furthermore, on $\delta\Sigma(\tau_p)$; $d\tau = 0$. Hence :-

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\right)_\tau \left(\frac{\partial x^\beta}{\partial y^b}\right)_\tau dy^a dy^b \\ &= g_{\alpha\beta} e^\alpha_a e^\beta_b dy^a dy^b \\ &= h_{ab} dy^a dy^b \end{aligned} \tag{5.59}$$

So, h_{ab} is a metric on $\delta\Sigma(\tau_p)$.

Now, $h = det[h_{ab}]$ and $\delta V = h^{\frac{1}{2}} d^3y$. Since, d^3y is constant on γ between two points p and q on γ; δV is due to $h^{\frac{1}{2}}$. So,

$$\frac{1}{\delta V}\frac{d(\delta V)}{d\tau} = \frac{1}{h^{\frac{1}{2}}}\frac{dh^{\frac{1}{2}}}{d\tau} = \frac{1}{2}\frac{1}{h}\frac{dh}{d\tau} = \frac{1}{2}h^{ab}\frac{dh_{ab}}{d\tau} \tag{5.60}$$

Now,

$$\begin{aligned}
\frac{dh_{ab}}{d\tau} &= u^\mu \nabla_\mu h_{ab} \\
&= u^\mu \nabla_\mu (g_{\alpha\beta} e^\alpha_a e^\beta_b) \\
&= u^\mu g_{\alpha\beta} (\nabla_\mu e^\alpha_a) e^\beta_b + u^\mu g_{\alpha\beta} (\nabla_\mu e^\beta_b) e^\alpha_a \\
&= g_{\alpha\beta} (e^\mu_a \nabla_\mu u^\alpha) e^\beta_b + g_{\alpha\beta} (e^\mu_b \nabla_\mu u^\beta) e^\alpha_a \\
&= e^\mu_a \nabla_\mu u_\beta e^\beta_b + e^\mu_b \nabla_\mu u_\alpha e^\alpha_a \\
&= e^\mu_a B_{\beta\mu} e^\beta_b + e^\mu_b B_{\alpha\mu} e^\alpha_a \\
&= (B_{\alpha\beta} + B_{\beta\alpha}) e^\alpha_a e^\beta_b \\
So,\ h^{ab} \frac{dh_{ab}}{d\tau} &= (B_{\alpha\beta} + B_{\beta\alpha}) h^{ab} e^\alpha_a e^\beta_b \\
&= (B_{\alpha\beta} + B_{\beta\alpha}) h^{\alpha\beta} \\
&= (B_{\alpha\beta} + B_{\beta\alpha}) g^{\alpha\beta}\ (on\ \gamma) \\
&= g^{\alpha\beta} B_{\alpha\beta} + g^{\beta\alpha} B_{\beta\alpha} \\
&= 2\theta
\end{aligned}$$

So,

$$\theta = \frac{1}{2} h^{ab} \frac{dh_{ab}}{d\tau} = \frac{1}{h^{\frac{1}{2}}} \frac{dh^{\frac{1}{2}}}{d\tau} = \frac{1}{\delta V} \frac{d(\delta V)}{d\tau} \tag{5.61}$$

So, θ is equal to the fractional rate of change of δV which is the congruence's cross-sectional volume. Hence, $\theta > 0$ means the geodesics are diverging and $\theta < 0$ means the geodesics are converging.

5.3.4 Raychaudhuri Equation for timelike geodesics

Consider,

$$\begin{aligned}
u^\mu \nabla_\mu B_{\alpha\beta} &= u^\mu \nabla_\mu \nabla_\beta u_\alpha \\
&= u^\mu (\nabla_\beta \nabla_\mu u_\alpha + R_{\mu\beta\alpha}{}^\nu u_\nu) \\
&= u^\mu (\nabla_\beta \nabla_\mu u_\alpha - R_{\beta\mu\alpha}{}^\nu u_\nu) \\
&= \nabla_\beta (u^\mu \nabla_\mu u_\alpha) - (\nabla_\beta u^\mu)(\nabla_\mu u_\alpha) - R_{\beta\mu\alpha}{}^\nu u_\nu u^\mu \\
&= -B^\mu_\beta - R_{\beta\mu\alpha}{}^\nu u_\nu u^\mu \\
Now,\ g^{\lambda\beta} &\rightarrow \\
u^\mu \nabla_\mu B^\lambda_\alpha &= -B^{\mu\lambda} B_{\alpha\mu} - R^\lambda_{\mu\alpha\nu} u^\nu u^\mu \\
tracing\ (\lambda = \alpha);\ u^\mu \nabla_\mu B^\alpha_\alpha &= -B^{\mu\alpha} B_{\alpha\mu} - R_{\mu\nu} u^\nu u^\mu \\
\Rightarrow\ \frac{d\theta}{d\tau} &= u^\mu \nabla_\mu B^\alpha_\alpha = -B^{\mu\alpha} B_{\alpha\mu} - R_{\mu\nu} u^\nu u^\mu \\
Now,\ B^{\mu\alpha} B_{\alpha\mu} &= \left(\frac{1}{3}\theta h^{\mu\alpha} + \sigma^{\mu\alpha} + \omega^{\mu\alpha}\right)\left(\frac{1}{3}\theta h_{\alpha\mu} + \sigma_{\alpha\mu} + \omega_{\alpha\mu}\right) \\
&= \frac{1}{3}\theta^2 + \sigma^{\alpha\mu}\sigma_{\alpha\mu} - \omega^{\alpha\mu}\omega_{\alpha\mu}
\end{aligned}$$

So, we finally obtain :-

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta} u^\alpha u^\beta \tag{5.62}$$

Eq^n 5.62 is known as the *Raychaudhuri Equation* for timelike geodesics.

Focusing theorem for timelike geodesics

Now consider timelike geodesics which are *hypersurface* orthogonal and obey the strong energy condition; $R_{\mu\nu} u^\mu u^\nu \geq 0$; then :-

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - R_{\alpha\beta} u^\alpha u^\beta \geq 0 \tag{5.63}$$

Eq^n 5.63 is known as the *focusing theorem* for timelike geodesics. So, we can see that if $\theta_i < 0$ then geodesics converge rapidly in future while if $\theta_i > 0$ then geodesics diverge less rapidly in future. This is in accordance with *Newtonian Gravity of Attraction*.

Now consider;

$$\begin{aligned} \frac{d\theta}{d\tau} &\leq -\frac{1}{3}\theta^2 \\ \Rightarrow \frac{d\theta}{\theta^2} &\leq -\frac{d\tau}{3} \\ Upon\ Integration;\ -(\theta^{-1} - \theta_i{}^{-1}) &\leq -\frac{\tau}{3} \\ \Rightarrow \frac{1}{\theta} &\geq \frac{1}{\theta_i} + \frac{\tau}{3} \end{aligned} \tag{5.64}$$

So, if $\theta_i < 0$ (initially converging geodesics) then $\theta(\tau) \to -\infty$ within $\tau \leq \frac{3}{|\theta_i|}$. So, within proper time $\tau \leq \frac{3}{|\theta_i|}$; geodesics form a *caustic* singularity (where some geodesics meet).

5.4 Congruence of Nulllike Geodesics

Definition 5.4.1. It is a family of non-intersecting nulllike geodesics on a manifold M such that their tangent vector fields are nulllike.

Let $g_{\alpha\beta}$ be the metric. k^α is the tangent vector to the geodesics and ξ^α is the deviation vector.

So,

$$k^\alpha k_\alpha = 0 \tag{5.65}$$
$$k^\beta \nabla_\beta \xi^\alpha = \xi^\beta \nabla_\beta k^\alpha \tag{5.66}$$
$$k^\beta \nabla_\beta k^\alpha = 0 \tag{5.67}$$
$$k^\alpha \xi_\alpha = 0 \tag{5.68}$$

5.4.1 Transverse Metric

Consider a null vector field N^α such that :-

$$k^\alpha N_\alpha = -1 \tag{5.69}$$
$$N^\alpha N_\alpha = 0 \tag{5.70}$$

Note. The above *cond*^{n}*s* on N^α do not give unique N^α.

Now, the transverse metric $h_{\alpha\beta}$ is given by :-

$$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta \tag{5.71}$$

Consider its orthogonality to k and N :-

$$h_{\alpha\beta}k^\beta = g_{\alpha\beta}k^\beta + k_\alpha N_\beta k^\beta + N_\alpha k_\beta k^\beta = k_\alpha - k_\alpha + 0 = 0 \tag{5.72}$$
$$k^\alpha h_{\alpha\beta} = k^\alpha g_{\alpha\beta} + k^\alpha k_\alpha N_\beta + k^\alpha N_\alpha k_\beta = k_\beta + 0 - k_\beta = 0 \tag{5.73}$$
$$h_{\alpha\beta}N^\beta = g_{\alpha\beta}N^\beta + k_\alpha N_\beta N^\beta + N_\alpha k_\beta N^\beta = N_\alpha - 0 - N_\alpha = 0 \tag{5.74}$$
$$N^\alpha h_{\alpha\beta} = N^\alpha g_{\alpha\beta} + N^\alpha k_\alpha N_\beta + N^\alpha N_\alpha k_\beta = N_\beta - N_\beta + 0 = 0 \tag{5.75}$$

Furthermore consider;

$$\begin{aligned} g^{\alpha\mu}h_{\alpha\beta} &= h^\mu_\beta = \delta^\mu_\beta + k^\mu N_\beta + N^\mu k_\beta \\ So,\ h^\mu_\mu &= \delta^\mu_\mu + k^\mu N_\mu + N^\mu k_\mu = 4 - 1 - 1 = 2 \\ Consider,\ h^\alpha_\mu h^\mu_\beta &= \delta^\alpha_\beta - k^\alpha N_\beta - N^\alpha k_\beta + k^\alpha N_\beta + N^\alpha k_\beta + k^\alpha N_\beta + k^\alpha N_\mu N^\mu k_\beta + N^\alpha k_\beta + N^\alpha k_\mu k^\mu N_\beta \\ &= \delta^\alpha_\beta + k^\alpha N_\beta + N^\alpha k_\beta = h^\alpha_\beta \end{aligned}$$

So, we get :-

$$h^{\alpha}_{\alpha} = 2 \tag{5.76}$$
$$h^{\alpha}_{\mu} h^{\mu}_{\beta} = h^{\alpha}_{\beta} \tag{5.77}$$

So, $h_{\alpha\beta}$ is indeed the transverse metric and the transverse space is $2-$dimensional.

Now define;

$$B_{\alpha\beta} = \nabla_{\beta} k_{\alpha} \tag{5.78}$$
$$Also,\ \tilde{\xi}^{\alpha} = h^{\alpha}_{\mu} \xi^{\mu} = \xi^{\alpha} + (N_{\mu}\xi^{\mu})k^{\alpha} \tag{5.79}$$

Now,

$$\begin{aligned}
k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha} &= k^{\beta}\nabla_{\beta}(h^{\alpha}_{\mu}\xi^{\mu}) \\
&= k^{\beta}\nabla_{\beta}[(\delta^{\alpha}_{\mu} + k^{\alpha}N_{\mu} + N^{\alpha}k_{\mu})\xi^{\mu}] \\
&= k^{\beta}\nabla_{\beta}[\xi^{\alpha} + k^{\alpha}N_{\mu}\xi^{\mu}] \\
&= k^{\beta}\nabla_{\beta}\xi^{\alpha} + k^{\beta}\nabla_{\beta}(k^{\alpha}N_{\mu}\xi^{\mu}) \\
&= k^{\beta}\nabla_{\beta}\xi^{\alpha} + k^{\beta}\nabla_{\beta}(N_{\mu}\xi^{\mu})k^{\alpha} \\
Also,\ k^{\beta}\nabla_{\beta}\xi^{\alpha} &= \xi^{\beta}\nabla_{\beta}k^{\alpha} = B^{\alpha}_{\beta}\xi^{\beta} && (5.80) \\
So,\ k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha} &= k^{\beta}\nabla_{\beta}(h^{\alpha}_{\mu}\xi^{\mu}) \\
&= h^{\alpha}_{\mu}B^{\mu}_{\beta}\xi^{\beta} + (k^{\beta}\nabla_{\beta}h^{\alpha}_{\mu})\xi^{\mu} \\
Now,\ k^{\beta}\nabla_{\beta}h^{\alpha}_{\mu} &= k^{\beta}\nabla_{\beta}(\delta^{\alpha}_{\mu} + k^{\alpha}N_{\mu} + N^{\alpha}k_{\mu}) \\
&= (k^{\beta}\nabla_{\beta}N_{\mu})k^{\alpha} + (k^{\beta}\nabla_{\beta}N^{\alpha})k_{\mu} \\
\Rightarrow\ k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha} &= h^{\alpha}_{\mu}B^{\mu}_{\beta}\xi^{\beta} + (k^{\beta}\nabla_{\beta}N_{\mu})k^{\alpha}\xi^{\mu} + (k^{\beta}\nabla_{\beta}N^{\alpha})k_{\mu}\xi^{\mu} \\
&= h^{\alpha}_{\mu}B^{\mu}_{\beta}\xi^{\beta} + (k^{\beta}\nabla_{\beta}N_{\mu})k^{\alpha}\xi^{\mu}\ (it\ has\ component\ along\ k^{\alpha}) && (5.81) \\
So,\ h^{\mu}_{\alpha}k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha} &= h^{\mu}_{\alpha}(h^{\alpha}_{\mu}B^{\mu}_{\beta}\xi^{\beta} + (k^{\beta}\nabla_{\beta}N_{\mu})k^{\alpha}\xi^{\mu}) \\
&= 2B^{\mu}_{\beta}\xi^{\beta} + h^{\mu}_{\alpha}(k^{\beta}\nabla_{\beta}N_{\mu})k^{\alpha}\xi^{\mu} \\
&= 2B^{\mu}_{\beta}\xi^{\beta} + (\delta^{\mu}_{\alpha} + k^{\mu}N_{\alpha} + N^{\mu}k_{\alpha})k^{\alpha}\xi^{\mu}(k^{\beta}\nabla_{\beta}N_{\mu}) \\
&= 2B^{\mu}_{\beta}\xi^{\beta} \\
&= h^{\mu}_{\alpha}(h^{\alpha}_{\nu}B^{\nu}_{\beta}\xi^{\beta}) = h^{\mu}_{\nu}B^{\nu}_{\beta}\xi^{\beta} && (5.82) \\
Now,\ B^{\nu}_{\beta}\tilde{\xi}^{\beta} &= B^{\nu}_{\beta}\xi^{\beta} + B^{\nu}_{\beta}(N_{\mu}\xi^{\mu})k^{\beta} \\
&= B^{\nu}_{\beta}\xi^{\beta} + (N_{\mu}\xi^{\mu})B^{\nu}_{\beta}k^{\beta} = B^{\nu}_{\beta}\xi^{\beta} && (5.83) \\
So,\ h^{\mu}_{\alpha}(k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha}) &= h^{\mu}_{\nu}B^{\nu}_{\beta}\tilde{\xi}^{\beta} \\
Now,\ h^{\alpha}_{\mu}\tilde{\xi}^{\mu} &= (\delta^{\alpha}_{\mu} + k^{\alpha}N_{\mu} + N^{\alpha}k_{\mu})(\xi^{\mu} + (N_{\beta}\xi^{\beta})k^{\mu}) \\
&= \xi^{\alpha} + (N_{\beta}\xi^{\beta})k^{\alpha} + k^{\alpha}N_{\mu}\xi^{\mu} = \tilde{\xi}^{\alpha} + (N_{\mu}\xi^{\mu})k^{\alpha} && (5.84) \\
Now,\ h^{\mu}_{\alpha}(k^{\beta}\nabla_{\beta}\tilde{\xi}^{\alpha}) &= h^{\mu}_{\nu}B^{\nu}_{\beta}\tilde{\xi}^{\beta} \\
&= h^{\mu}_{\nu}B^{\nu}_{\beta}(h^{\beta}_{\lambda}\tilde{\xi}^{\lambda} - (N_{\mu}\xi^{\mu})k^{\beta}) \\
&= h^{\mu}_{\nu}h^{\beta}_{\lambda}B^{\nu}_{\beta}\tilde{\xi}^{\lambda} && (5.85) \\
So,\ (k^{\beta}\nabla_{\beta}\tilde{\xi}^{\mu})' &= h^{\mu}_{\nu}h^{\beta}_{\lambda}B^{\nu}_{\beta}\tilde{\xi}^{\lambda} = \tilde{B}^{\mu}_{\lambda}\tilde{\xi}^{\lambda} && (5.86) \\
where,\ \tilde{B}^{\mu}_{\lambda} &= h^{\mu}_{\nu}h^{\beta}_{\lambda}B^{\nu}_{\beta} \\
Multiplying\ by\ g_{\mu\nu} \rightarrow \tilde{B}_{\alpha\beta} &= h^{\mu}_{\alpha}h^{\nu}_{\beta}B_{\mu\nu} && (5.87) \\
Now,\ \tilde{B}_{\alpha\beta} &= (\delta^{\mu}_{\alpha} + k^{\mu}N_{\alpha} + N^{\mu}k_{\alpha})(\delta^{\nu}_{\beta} + k^{\nu}N_{\beta} + N^{\nu}k_{\beta})B_{\mu\nu} \\
&= (\delta^{\mu}_{\alpha} + k^{\mu}N_{\alpha} + N^{\mu}k_{\alpha})(B_{\mu\beta} + k_{\beta}B_{\mu\nu}N^{\nu}) \\
&= B_{\alpha\beta} + k_{\beta}B_{\alpha\nu}N^{\nu} + k_{\alpha}N^{\mu}B_{\mu\beta} + k_{\alpha}k_{\beta}B_{\mu\nu}N^{\mu}N^{\nu} && (5.88)
\end{aligned}$$

Furthermore, it is really straightforward to check that :-

$$\tilde{B}_{\alpha\beta}k^{\beta} = k^{\alpha}\tilde{B}_{\alpha\beta} = 0 \tag{5.89}$$
$$\tilde{B}_{\alpha\beta}N^{\beta} = N^{\alpha}\tilde{B}_{\alpha\beta} = 0 \tag{5.90}$$

So, $\tilde{B}_{\alpha\beta}$ is purely transversal as we desired. Now,

$$\begin{aligned}
\tilde{B}_{\alpha\beta} &= \frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \\
Now,\ \tilde{B}^{\alpha}_{\beta} &= \frac{1}{2}\theta h^{\alpha}_{\beta} + \sigma^{\alpha}_{\beta} + \omega^{\alpha}_{\beta} \\
So,\ \tilde{B}^{\alpha}_{\alpha} &= \frac{1}{2}\theta h^{\alpha}_{\alpha} = \theta \\
Hence,\ \theta &= g^{\alpha\beta}\tilde{B}_{\alpha\beta} \\
&= g^{\alpha\beta}(B_{\alpha\beta} + k_\beta B_{\alpha\nu}N^\nu + k_\alpha N^\mu B_{\mu\beta} + k_\alpha k_\beta B_{\mu\nu}N^\mu N^\nu) \\
&= g^{\alpha\beta}B_{\alpha\beta} = g^{\alpha\beta}(\nabla_\beta k_\alpha) = \nabla_\beta k^\beta \ (independent\ of\ choice\ of\ N^\mu) \qquad (5.91)
\end{aligned}$$

5.4.2 Frobenius' Theorem for nulllike geodesics

The general version of the *Frobenius' Theorem* holds here too (as no assumption was made on the nature of tangent vector on the first part of the proof), i.e.;

$$hypersurface\ orthogonal \ \Leftrightarrow\ k_{[\alpha}\nabla_\beta k_{\gamma]} = 0 \qquad (5.92)$$

Now,

$$\begin{aligned}
k_{[\alpha}\nabla_\beta k_{\gamma]} &= 2[k_\alpha B_{[\gamma\beta]} + k_\gamma B_{[\beta\alpha]} + k_\beta B_{[\alpha\gamma]}] = 0 \\
&\Rightarrow\ k_\alpha B_{[\gamma\beta]} + k_\gamma B_{[\beta\alpha]} + k_\beta B_{[\alpha\gamma]} = 0 \qquad (5.93)
\end{aligned}$$

Now,

$$\begin{aligned}
B_{[\alpha\beta]}k_\gamma + B_{[\gamma\alpha]}k_\beta + B_{[\beta\gamma]}k_\alpha &= 0 \\
So,\ B_{[\alpha\beta]}k_\gamma N^\gamma + B_{[\gamma\alpha]}k_\beta N^\gamma + B_{[\beta\gamma]}k_\alpha N^\gamma &= 0
\end{aligned}$$

Furthermore,

$$\begin{aligned}
B_{[\alpha\beta]} &= B_{[\gamma\alpha]}k_\beta N^\gamma + B_{[\beta\gamma]}k_\alpha N^\gamma \\
&= \left(\frac{1}{2}(B_{\gamma\alpha} - B_{\alpha\gamma})k_\beta + \frac{1}{2}(B_{\beta\gamma} - B_{\gamma\beta}k_\alpha)\right)N^\gamma \\
&= B_{\gamma[\alpha}k_{\beta]}N^\gamma + k_{[\alpha}B_{\beta]\gamma}N^\gamma \qquad (5.94)
\end{aligned}$$

Now,

$$\begin{aligned}
\tilde{B}_{[\alpha\beta]} &= \frac{1}{2}(\tilde{B}_{\alpha\beta} - \tilde{B}_{\beta\alpha}) \\
&= \frac{1}{2}(B_{\alpha\beta} + k_\beta B_{\alpha\nu}N^\nu + k_\alpha N^\mu B_{\mu\beta} + k_\alpha k_\beta B_{\mu\nu}N^\mu N^\nu - B_{\beta\alpha} - k_\beta N^\mu B_{\mu\alpha} - k_\beta k_\alpha B_{\mu\nu}N^\mu N^\nu - k_\alpha B_{\beta\nu}N^\nu) \\
&= B_{[\alpha\beta]} - N^\mu B_{\mu[\alpha}k_{\beta]} - k_{[\alpha}B_{\beta]\nu}N^\nu \\
&= 0\ (by\ Eq^n\ 5.94)
\end{aligned}$$

So, we finally get :-

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]} = 0 \qquad (5.95)$$

So, finally we state *Frobenius theorem* for nulllike geodesics :-

Theorem 5.4.1. *Let γs be nulllike geodesics which are hypersurface orthogonal with tangent vector as k. Furthermore, let ω be the rotation tensor. Then,*

$$hypersurface\ orthogonal \ \Leftrightarrow\ k_{[\alpha}\nabla_\beta k_{\gamma]} = 0 \ \Rightarrow\ \omega_{\alpha\beta} = 0 \qquad (5.96)$$

Remark. Here the *hypersurfaces* considered are null *hypersurfaces.*

5.4.3 Interpretation of θ for nulllike geodesics

We proceed by the same consideration as was done in the timelike case. Only difference is that the transverse space here is 2−dimensional. Let us pick a particular geodesic γ from the null congruence, and on this geodesic we select a point p at which $\lambda = \lambda_p$ for some $\lambda \epsilon \Re$. We then consider the null curves to which N^α is tangent, and we let $\mu \epsilon \Re$ be the parameter on these auxiliary curves; we adjust the parameterization so that μ is constant on the null geodesics. The auxiliary curve that passes through p is called β, and we have that $\mu = \mu_\gamma$ at p. The cross-section $\delta S(\lambda_p)$ is defined to be a small set of points $p^{'}$ in a *nbd* of p such that :-
a) At each point $p^{'}$, $\lambda = \lambda_p$ and $\mu = \mu_\gamma$.
b) Through different points $p^{'} \epsilon \delta S(\lambda_p)$ different auxiliary curves different geodesics pass through from the given null congruence.

This set forms a two-dimensional region, the intersection of small segments of the *hypersurfaces* $\lambda = \lambda_p$ and $\mu = \mu_\gamma$. We assume that the parameterization has been adjusted so that both γ and β intersect $\delta S(\lambda_p)$ orthogonally (no requirements on other curves).

We introduce coordinates in $\delta S(\lambda_p)$ by assigning a label θ^A where, $(A \epsilon \{2, 3\}$ to each point in the set. Recalling that through each of these points there passes a geodesic from the congruence, we see that we may use θ^A to label the geodesics themselves. By demanding that each geodesic keep its label as it moves away from $\delta S(\lambda_p)$, we simultaneously obtain a coordinate system θ^A in any other cross-section $\delta S(\lambda_p)$. This construction therefore produces a coordinate system (λ, μ, θ^A) in a *nbd* of the geodesic γ, and $\exists$ a transformation between this system and the one originally in use :-

$$x^\alpha = x^\alpha(\lambda, \mu, \theta^A) \tag{5.97}$$

Because μ and θ^A are constant along the geodesics, we have :-

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\mu,\theta^A} \quad (tangent\ vector\ field) \tag{5.98}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_{\lambda,\mu} \quad (transverse\ deviation\ vector) \tag{5.99}$$

Furthermore as before,

$$k^\mu \nabla_\mu e^\alpha_A = e^\mu_A \nabla_\mu k^\alpha \tag{5.100}$$

$$And,\ k_\alpha e^\alpha_A = 0 = N_\alpha e^\alpha_A\ (by\ parametrization) \tag{5.101}$$

$$Also,\ \pounds_k e^\alpha_A = 0 \tag{5.102}$$

Now we define a 2− tensor as which as before would be a metric on $\delta S(\lambda_p)$:-

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B \tag{5.103}$$

Also, as before :-

$$\sigma_{AB} = h_{\alpha\beta} e^\alpha_A e^\beta_B \tag{5.104}$$

$$h^{\alpha\beta} = \sigma^{AB} e^\alpha_A e^\beta_B \tag{5.105}$$

Now, $\sigma = det[\sigma_{AB}]$ and $\delta A = \sigma^{\frac{1}{2}} d^2\theta^A$. Since, $d^2\theta^A$ is constant on γ between two points p and q on γ; δA is due to $\sigma^{\frac{1}{2}}$. So,

$$\frac{1}{\delta A}\frac{d(\delta A)}{d\lambda} = \frac{1}{\sigma^{\frac{1}{2}}}\frac{d\sigma^{\frac{1}{2}}}{d\lambda} = \frac{1}{2}\frac{1}{\sigma}\frac{d\sigma}{d\lambda} = \frac{1}{2}\sigma^{AB}\frac{d\sigma_{AB}}{d\lambda} \tag{5.106}$$

Now,

$$\begin{aligned}
\frac{d\sigma_{AB}}{d\lambda} &= k^\mu \nabla_\mu \sigma_{AB} \\
&= k^\mu \nabla_\mu (g_{\alpha\beta} e^\alpha_A e^\beta_B) \\
&= g_{\alpha\beta}(e^\mu_A \nabla_\mu k^\alpha) e^\beta_B + g_{\alpha\beta}(e^\mu_B \nabla_\mu k^\beta) e^\alpha_A \\
&= e^\mu_A B_{\beta\mu} e^\beta_B + e^\mu_B B_{\alpha\mu} e^\alpha_A \\
&= e^\alpha_A e^\beta_B (B_{\alpha\beta} + B_{\beta\alpha})
\end{aligned}$$

So,

$$\begin{aligned}\sigma^{AB}\frac{d\sigma_{AB}}{d\lambda} &= \sigma^{AB}e^\alpha_A e^\beta_B(B_{\alpha\beta}+B_{\beta\alpha})\\ &= g^{\alpha\beta}B_{\alpha\beta}+g^{\beta\alpha}B_{\beta\alpha}\ (on\ \gamma)\\ &= g^{\alpha\beta}\tilde{B}_{\alpha\beta}+g^{\beta\alpha}\tilde{B}_{\beta\alpha}\\ &= 2\theta\end{aligned}$$

So,

$$\theta = \frac{1}{2}\sigma^{AB}\frac{d\sigma_{AB}}{d\lambda} = \frac{1}{\sigma^{\frac{1}{2}}}\frac{d\sigma^{\frac{1}{2}}}{d\lambda} = \frac{1}{\delta A}\frac{d(\delta A)}{d\lambda} \tag{5.107}$$

So, θ is equal to the fractional rate of change of δA which is the congruence's cross-sectional area. Hence, $\theta > 0$ means the null geodesics are diverging and $\theta < 0$ means the null geodesics are converging.

5.4.4 Raychaudhuri Equation for nulllike geodesics

We know,

$$\begin{aligned}T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T &= \frac{R_{\mu\nu}}{8\pi}\\ So,\ T_{\mu\nu}k^\mu k^\nu-\frac{1}{2}g_{\mu\nu}Tk^\mu k^\nu &= \frac{R_{\mu\nu}}{k}^{\mu}k^\nu 8\pi\\ Hence,\ R_{\mu\nu}k^\mu k^\nu &= 8\pi T_{\mu\nu}k^\mu k^\nu\\ Now,\ T_{\mu\nu}k^\mu k^\nu \geq 0 &\Rightarrow\ R_{\mu\nu}k^\mu k^\nu \geq 0\end{aligned}$$

Now consider;

$$\begin{aligned}-\tilde{B}^{\mu\alpha}\tilde{B}_{\alpha\mu} &= -(B^{\mu\alpha}+k^\mu N^\gamma B^\alpha_\gamma+k^\alpha B^\mu_\gamma N^\gamma+k^\mu k^\alpha B_{\gamma\nu}N^\gamma N^\nu)(B_{\alpha\mu}+k_\alpha N^\gamma B_{\gamma\mu}+k_\mu B_{\alpha\gamma}N^\gamma+k_\alpha k_\mu B_{\gamma\nu}N^\gamma N^\nu)\\ &= -B^{\mu\alpha}B_{\alpha\mu}\end{aligned}$$

Now, as before;

$$B^{\mu\alpha}B_{\alpha\mu} = \tilde{B}^{\mu\alpha}\tilde{B}_{\alpha\mu} = \frac{1}{2}\theta^2+\sigma^{\alpha\mu}\sigma_{\alpha\mu}-\omega^{\alpha\mu}\omega_{\alpha\mu} \tag{5.108}$$

Now, as before we see that :-

$$\frac{d\theta}{d\lambda} = -B^{\mu\alpha}B_{\alpha\mu}-R_{\mu\nu}k^\mu k^\nu \tag{5.109}$$

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2-\sigma^{\alpha\mu}\sigma_{\alpha\mu}+\omega^{\alpha\mu}\omega_{\alpha\mu}-R_{\mu\nu}k^\mu k^\nu \tag{5.110}$$

Eq^n 5.110 is known as the *Raychaudhuri Equation* for nulllike geodesics.

Focusing theorem for nulllike geodesics

Now consider nulllike geodesics which are *hypersurface* orthogonal and obey the null energy condition; $R_{\mu\nu}u^\mu u^\nu \geq 0$; then :-

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2-\sigma^{\alpha\beta}\sigma_{\alpha\beta}-R_{\alpha\beta}k^\alpha k^\beta \geq 0 \tag{5.111}$$

Eq^n 5.111 is known as the *focusing theorem* for nulllike geodesics. So, we can see that if $\theta_i < 0$ then geodesics converge rapidly in future while if $\theta_i > 0$ then geodesics diverge less rapidly in future. This is in accordance with *Newtonian Gravity of Attraction.*

Now consider;

$$\begin{aligned}\frac{d\theta}{d\lambda} &\leq -\frac{1}{2}\theta^2\\ \Rightarrow \frac{d\theta}{\theta^2} &\leq -\frac{d\lambda}{2}\\ Upon\ Integration;\ -(\theta^{-1}-\theta_i{}^{-1}) &\leq -\frac{\lambda}{2}\\ \Rightarrow \frac{1}{\theta} &\geq \frac{1}{\theta_i}+\frac{\lambda}{2}\end{aligned} \tag{5.112}$$

So, if $\theta_i < 0$ (initially converging geodesics) then $\theta(\lambda) \to -\infty$ within $\lambda \leq \frac{2}{|\theta_i|}$. So, within $\lambda \leq \frac{2}{|\theta_i|}$; null geodesics form a *caustic* singularity (where some null geodesics meet).

5.4.5 Some more properties of null congruence

Consider,

$$\begin{aligned}\tilde{B}_{(\mu\nu)} &= \frac{1}{2}(\tilde{B}_{\mu\nu} + \tilde{B}_{\nu\mu}) \\ &= \frac{1}{2}\left(\frac{1}{2}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{2}\theta h_{\beta\alpha} + \sigma_{\beta\alpha} + \omega_{\beta\alpha}\right) \\ &= \frac{1}{2}\left(2\sigma_{\alpha\beta} + \theta h_{\alpha\beta}\right) \\ &= \sigma_{\alpha\beta} + \frac{1}{2}\theta h_{\alpha\beta}\end{aligned}$$

Now for null geodesics which are *hypersurface* orthogonal; $\omega_{\alpha\beta} = 0$. So,

$$\begin{aligned}\tilde{B}_{\alpha\beta} &= \tilde{B}_{(\alpha\beta)} = h^{\mu}_{\alpha}h^{\nu}_{\beta}B_{\mu\nu} \\ \tilde{B}_{\beta\alpha} &= h^{\mu}_{\alpha}h^{\nu}_{\beta}B_{\nu\mu} \\ So,\ \frac{1}{2}(\tilde{B}_{\alpha\beta} + \tilde{B}_{\beta\alpha}) &= \frac{1}{2}h^{\mu}_{\alpha}h^{\nu}_{\beta}(B_{\mu\nu} + B_{\nu\mu}) \\ So,\ \tilde{B}_{(\alpha\beta)} &= h^{\mu}_{\alpha}h^{\nu}_{\beta}\tilde{B}_{(\mu\nu)} \qquad (5.113)\end{aligned}$$

Hence,

$$\begin{aligned}\tilde{B}_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} &= h^{\mu}_{\alpha}\tilde{B}_{(\mu\nu)}h^{\nu}_{\beta} \qquad (5.114) \\ &= h^{\mu}_{\alpha}\nabla_{(\nu}k_{\mu)}h^{\nu}_{\beta} \qquad (5.115)\end{aligned}$$

Now, let us suppose $\mathcal{N}$ is a *killing horizon* and ξ is a *killing vector field.* So, for some $f\epsilon\mathcal{F}$; $\xi = fk$ or, $k = f^{-1}\xi$.

So,

$$\begin{aligned}\tilde{B}_{\alpha\beta} &= h^{\mu}_{\alpha}\nabla_{(\nu}f^{-1}\xi_{\mu)}h^{\nu}_{\beta} \\ &= h^{\mu}_{\alpha}[\partial_{(\nu}f^{-1}\xi_{\mu)} + f^{-1}\nabla_{(\nu}\xi_{\mu)}]h^{\nu}_{\beta} \\ &= h^{\mu}_{\alpha}[\partial_{(\nu}f^{-1}\xi_{\mu)}]h^{\nu}_{\beta}\ (as\ \nabla_{\nu}\xi_{\mu}\ is\ completely\ antisymmetric)\end{aligned}$$

Now we know;

$$h_{\alpha\beta}\xi^{\beta} = 0 = \xi^{\alpha}h_{\alpha\beta} \qquad (5.116)$$

So, we get $\tilde{B}_{\alpha\beta} = 0$ on $\mathcal{N}$. Furthermore, since $\omega_{\alpha\beta} = 0$ on $\mathcal{N}$; we obtain from expression of $\tilde{B}_{\alpha\beta}$:-

$$\begin{aligned}\frac{1}{2}\theta h_{\alpha\beta} &= -\sigma_{\alpha\beta} \\ But,\ \theta = g^{\alpha\beta}\tilde{B}_{\alpha\beta} &= 0\ (on\ \mathcal{N}) \qquad (5.117) \\ Hence,\ \sigma_{\alpha\beta} &= 0\ (on\ \mathcal{N}) \qquad (5.118)\end{aligned}$$

So, finally we get :-

$$\begin{aligned}\left|\frac{d\theta}{d\lambda}\right|_{\mathcal{N}} &= 0\ (as\ \theta = 0\ on\ \mathcal{N}) \qquad (5.119) \\ Now\ by\ Raychaudhuri\ Eq^n\ 5.110;\ |R_{\mu\nu}k^{\mu}k^{\nu}|_{\mathcal{N}} &= 0 \qquad (5.120) \\ This\ also\ gives;\ |T_{\mu\nu}k^{\mu}k^{\nu}|_{\mathcal{N}} &= 0 \qquad (5.121)\end{aligned}$$

Chapter 6

Black Holes and Black Rings

We won't go into details of defining a black hole but just for the sake of defining we would say that, *"any finite solution of Einstein's Field Equations with a horizon (which is essentially a null hypersurface) is a black hole"*. The most important feature of a black hole spacetime is the event horizon, a null *hypersurface* which acts as a causal boundary between two regions of the spacetime, the interior and exterior of the black hole.

A spacetime containing a black hole possesses two distinct regions, the interior and exterior of the black hole; they are distinguished by the property that all external observers are causally disconnected from events occurring inside. Physically speaking, this corresponds to the fact that once one has entered a black hole, an observer can no longer send signals to the outside world.

Black hole singularity to be considered here will be defined later.

6.1 $4-$dimensional Black Holes

Here we will present some stationary black hole solutions in vacuum in a $4-$dimensional spacetime manifold.

6.1.1 Schwarzschild Black Hole

The Schwarzschild metric, which is the spacetime solution of *Einstein's Field Equations* for a point mass M, in polar coordinates (t, r, θ, ϕ) is given by :-

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the S^2 metric. Coordinate singularity is at $g^{rr} = 0$, i.e.; $r = 2M$. True singularity/singularity is at $r = 0$.

Event Horizon

Now, let us examine the family of *hypersurfaces* $r = constant$ denoted by Σ. From this family let us fix our discussion on $r = 2M$. Now, examine the Schwarzschild metric :-
For $r > 2M$; $r-$coordinate is spacelike.
For $0 < r < 2M$; $r-$coordinate is timelike.

Hence, for $r < 2M$ region; causality allows r to be unidirectional and hence ingoing observer can't escape the singularity. Also, they can receive outside signal but can't send signal (due to maximum speed limit c).

Now, let us compute the normal vector field for $r = 2M$ *hypersurface* :-

$$\begin{aligned}
l &= \tilde{f} g^{\mu\nu} \frac{\partial r}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\
&= \tilde{f} g^{rr} \frac{\partial}{\partial r} \\
So,\ l^\mu &= l^r = \tilde{f} g^{rr} \\
Now,\ l_\mu &= g_{\mu\nu} l^\nu = g_{rr} l^r = g_{rr} \tilde{f} g^{rr} = \tilde{f} \\
Finally,\ |l^\mu l_\mu|_{r=2M} &= |l^r l_r|_{r=2M} = \left|\tilde{f}^2 g^{rr}\right|_{r=2M} = 0 \ (as\ g^{rr} = 0\ at\ r = 2M)
\end{aligned} \tag{6.2}$$

So, $r = 2M$ is a null *hypersurface*. From above consideration we conclude that $r = 2M$ is indeed the event horizon and the family of *hypersurfaces* Σ contains the event horizon.

6.1.2 Reissner-Nordström Black Hole

The Reissner-Nordström metric, which is the spacetime solution of *Einstein's Field Equations* for a point mass M with charge Q, in polar coordinates (t, r, θ, ϕ) is given by :-

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6.3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the S^2 metric. Coordinate singularities are at $g^{rr} = 0$, i.e.; $r = r_\pm = M \pm (M^2 - Q^2)^{\frac{1}{2}}$. True singularity/singularity is at $r = 0$.

Event Horizon

Now, let us examine the family of *hypersurfaces* $r = constant$ denoted by Σ. From this family let us fix our discussion on $r = r_\pm$. Now, examine the Reissner-Nordström metric :-
For $r > r_+$; $r-$coordinate is spacelike.
For $r_- < r < r_+$; $r-$coordinate is timelike.
For $0 < r < r_-$; $r-$coordinate is again spacelike.

Now observe here that for $r_- < r < r_+$ region; causality allows r to be unidirectional and ingoing observes would definitely enter into the $0 < r < r_-$ region. But something really interesting happens in the region $0 < r < r_-$ where $r-$coordinate is again spacelike and so the ingoing observer has a choice whether to go towards the singularity at $r = 0$ or go away from it. This is something like the notion of *repulsive gravity* and is mediated by what is known as the *black hole tunnels* but when the ingoing observer comes out moving away from the singularity then he reaches an external spacetime different from where he had started entering towards the black hole.

Now, let us compute the normal vector field for $r = r_\pm$ *hypersurface* :-

$$\begin{aligned}
l &= \tilde{f} g^{\mu\nu} \frac{\partial r}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\
&= \tilde{f} g^{rr} \frac{\partial}{\partial r} \\
So,\ l^\mu &= l^r = \tilde{f} g^{rr} \\
Now,\ l_\mu &= g_{\mu\nu} l^\nu = g_{rr} l^r = g_{rr} \tilde{f} g^{rr} = \tilde{f} \\
Finally,\ |l^\mu l_\mu|_{r=r_\pm} &= |l^r l_r|_{r=r_\pm} = \left|\tilde{f}^2 g^{rr}\right|_{r=r_\pm} = 0\ (as\ g^{rr} = 0\ at\ r = r_\pm)
\end{aligned} \tag{6.4}$$

So, $r = r_\pm$ are null *hypersurfaces*. $r = r_+$ is termed as the outer horizon and $r = r_-$ is termed as the inner horizon. From above consideration we conclude that $r = r_+$ is indeed the event horizon and the family of *hypersurfaces* Σ contains the event horizon.

If $|Q| = M$ then both the horizons coincide and the Reissner-Nordström black hole obtained is known as an *extreme* Reissner-Nordström black hole. If $|Q| > M$, then the Reissner-Nordström metric describes a *naked singularity* at $r = 0$.

6.1.3 Kerr Black Hole

The Kerr metric, which is the spacetime solution of *Einstein's Field Equations* for a rotating point mass M with angular momentum J, in *Boyer-Lindquist* coordinates (t, r, θ, ϕ) is given by :-

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{6.5}$$

$$= -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{6.6}$$

where;

$$a \equiv \frac{J}{M} \tag{6.7}$$
$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{6.8}$$
$$\Delta = r^2 - 2Mr + a^2 \tag{6.9}$$
$$\Sigma = (r^2 + a^2)^2 - a^2\Delta \sin^2\theta \tag{6.10}$$
$$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Sigma} \tag{6.11}$$

Coordinate singularities are at $g^{rr} = 0$, i.e. at $\Delta = 0$, which means at; $r = r_\pm = M \pm (M^2 - a^2)^{\frac{1}{2}}$. True singularity/singularity is at $r = 0$.

Now consider, with τ as the proper time parametrization;

$$L \equiv u^\alpha \phi_\alpha = 0 \ (observer\ with\ zero\ angular\ momentum) \tag{6.12}$$
$$\Rightarrow g_{\mu\nu}u^\mu \phi^\nu = 0$$
$$\Rightarrow g_{tt}\frac{dt}{d\tau}\frac{\partial t}{\partial\phi} + g_{t\phi}\frac{dt}{d\tau}\frac{\partial\phi}{\partial\phi} + g_{\phi\phi}\frac{d\phi}{d\tau}\frac{\partial\phi}{\partial\phi} = 0$$
$$\Rightarrow g_{t\phi}\dot{t} + g_{\phi\phi}\dot{\phi} = 0 \tag{6.13}$$
$$\Rightarrow \Omega \equiv \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \omega \ (dimension\ of\ angular\ velocity) \tag{6.14}$$

Event Horizon

Now, let us examine the family of *hypersurfaces* $r = constant$ denoted by S. From this family let us fix our discussion on $r = r_\pm$. Now, examine the Kerr metric :-
For $r > r_+$; $r-$coordinate is spacelike.
For $r_- < r < r_+$; $r-$coordinate is timelike.
For $0 < r < r_-$; $r-$coordinate is again spacelike.

Now observe here that for $r_- < r < r_+$ region; causality allows r to be unidirectional and ingoing observes would definitely enter into the $0 < r < r_-$ region. Again like previously, in the region $0 < r < r_-$ where $r-$coordinate is again spacelike and so the ingoing observer has a choice whether to go towards the singularity at $r = 0$ or go away from it.

Now, let us compute the normal vector field for $r = r_\pm$ *hypersurface* :-

$$l = \tilde{f} g^{\mu\nu}\frac{\partial r}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$$
$$= \tilde{f} g^{rr}\frac{\partial}{\partial r}$$
$$So,\ l^\mu = l^r = \tilde{f} g^{rr}$$
$$Now,\ l_\mu = g_{\mu\nu}l^\nu = g_{rr}l^r = g_{rr}\tilde{f} g^{rr} = \tilde{f}$$
$$Finally,\ |l^\mu l_\mu|_{r=r_\pm} = |l^r l_r|_{r=r_\pm} = \left|\tilde{f}^2 g^{rr}\right|_{r=r_\pm} = 0 \ (as\ g^{rr} = 0\ at\ r = r_\pm) \tag{6.15}$$

So, $r = r_\pm$ are null *hypersurfaces*. $r = r_+$ is termed as the outer horizon and $r = r_-$ is termed as the inner horizon. From above consideration we conclude that $r = r_+$ is indeed the event horizon and the family of *hypersurfaces* S contains the event horizon.

If $|a| = M$ then both the horizons coincide and the Kerr black hole obtained is known as an *extreme* Kerr black hole. If $|a| > M$, then the Kerr metric describes a *naked singularity* at $r = 0$.

6.1.4 Kerr-Newman Black Hole

This has exactly the same computations as that of the Kerr black hole. The Kerr-Newman metric, which is the spacetime solution of *Einstein's Field Equations* for a rotating point mass M with angular momentum J and also with charge Q, in *Boyer-Lindquist* coordinates (t, r, θ, ϕ) is given by :-

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{6.16}$$

where;

$$a \equiv \frac{J}{M} \tag{6.17}$$
$$\rho^2 = r^2 + a^2\cos^2\theta \tag{6.18}$$
$$\Delta = r^2 - 2Mr + a^2 + Q^2 \tag{6.19}$$
$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \tag{6.20}$$
$$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a(r^2 + a^2 - \Delta)}{\Sigma} \tag{6.21}$$

Coordinate singularities are at $g^{rr} = 0$, i.e. at $\Delta = 0$, which means at; $r = r_\pm = M \pm (M^2 - a^2 - Q^2)^{\frac{1}{2}}$. True singularity/singularity is at $r = 0$.

Event Horizon

Now, let us examine the family of *hypersurfaces* $r = constant$ denoted by S. From this family let us fix our discussion on $r = r_\pm$. Now, examine the Kerr-Newman metric :-
For $r > r_+$; $r-$coordinate is spacelike.
For $r_- < r < r_+$; $r-$coordinate is timelike.
For $0 < r < r_-$; $r-$coordinate is again spacelike.

Now observe here that for $r_- < r < r_+$ region; causality allows r to be unidirectional and ingoing observes would definitely enter into the $0 < r < r_-$ region. Again like previously, in the region $0 < r < r_-$ where $r-$coordinate is again spacelike and so the ingoing observer has a choice whether to go towards the singularity at $r = 0$ or go away from it.

Now, let us compute the normal vector field for $r = r_\pm$ *hypersurface* :-

$$\begin{aligned}
l &= \tilde{f}g^{\mu\nu}\frac{\partial r}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \\
&= \tilde{f}g^{rr}\frac{\partial}{\partial r} \\
So,\ l^\mu &= l^r = \tilde{f}g^{rr} \\
Now,\ l_\mu &= g_{\mu\nu}l^\nu = g_{rr}l^r = g_{rr}\tilde{f}g^{rr} = \tilde{f} \\
Finally,\ |l^\mu l_\mu|_{r=r_\pm} &= |l^r l_r|_{r=r_\pm} = \left|\tilde{f}^2 g^{rr}\right|_{r=r_\pm} = 0\ (as\ g^{rr} = 0\ at\ r = r_\pm)
\end{aligned} \tag{6.22}$$

So, $r = r_\pm$ are null *hypersurfaces*. $r = r_+$ is termed as the outer horizon and $r = r_-$ is termed as the inner horizon. From above consideration we conclude that $r = r_+$ is indeed the event horizon and the family of *hypersurfaces* S contains the event horizon.

If $a^2 + Q^2 = M^2$ then both the horizons coincide and the Kerr-Newman black hole obtained is known as an *extreme* Kerr-Newman black hole. If $a^2 + Q^2 > M^2$, then the Kerr-Newman metric describes a *naked singularity* at $r = 0$.

6.2 Horizons of Black Holes

Here we do some discussions on topology of horizons of $4-$dimensional black holes and the concept of *Bifurcation 2-sphere* along with the notion of singularities of black holes.

6.2.1 Topology of Black Holes

In this discussion we restrict ourselves to studying the horizons of Schwarzschild and Reissner-Nordström black hole and without proof generalise the result to all stationary $4-$dimensional black holes. Both the Schwarzschild and Reissner-Nordström metrics can be written in the form :-

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{6.23}$$

where $f(r)$ is a smooth real valued function of r. Now, observe the metric for $r = constant$ hypersurfaces. This gives :-

$$ds^2 = -f(r)dt^2 + r^2 d\Omega^2 \tag{6.24}$$

Now, perform time slicing of this metric, i.e.; consider $t = constant$ cross-sections of it. This gives :-

$$ds^2 = r^2 d\Omega^2 \tag{6.25}$$

which is nothing but the metric for S^2. Hence, the topology of all stationary $4-$dimensional black holes is spherical, i.e; same as that of S^2.

Bifurcation $2-$sphere

If the event horizon would be a *killing horizon* of the *killing vector field* of the spacetime then since it has spherical topology $\exists$ a $2-$sphere about the singularity $r = 0$; where the *killing vector field* vanishes. This 2-sphere is known as the *Bifurcation 2-sphere.*

6.2.2 Singularities

A singularity of the metric is a point at which the determinant of either it or its inverse vanishes. However, a singularity of the metric may be simply due to a failure of the coordinate system. If no coordinate system exists for which the singularity is removable then it is irremovable, i.e. a genuine singularity of the spacetime. Any singularity for which some scalar constructed from the *Riemann-Christoffel Curvature Tensor* blows up as it is approached is irremovable. Such singularities are called *curvature singularities.* It is virtually always true that the existence of a singularity as just defined can be detected by the incompleteness of some geodesic, i.e. there is some geodesic that cannot be continued to all values of its affine parameter. For this reason, we shall be defining a spacetime singularity in terms of *geodesic incompleteness.* Thus, a spacetime is *non-singular* iff all geodesics can be extended to all values of their affine parameters, changing coordinates if necessary. Furthermore, *geodesically complete* black holes will always contain a *Bifurcation 2-sphere.*

6.3 The Zeroth Law

Let us first state the Zeroth Law of Black Hole Mechanics :-

Theorem 6.3.1. *"Surface gravity κ of a stationary black hole is uniform over the entire event horizon."*

Proof. *The proof will be done in two steps. First we will prove that κ is constant along horizon's null generators. Then, we would prove that κ is constant from generator to generator.*

Let $\mathcal{N}$ be the null hypersurface which is also a killing horizon to a killing vector field ξ of the spacetime. Here we introduce the same familiar coordinates on event horizon cross-sections as was considered in the null congruence's cross-section, i.e; (v, μ, θ^A) (where v is the geodesic parameter); with same definition of tangent vector field and deviation vector applying here too. Now,

$$\begin{aligned}
\kappa^2 &= -\frac{1}{2}\left|(\nabla^\alpha \xi^\beta)(\nabla_\alpha \xi_\beta)\right|_{\mathcal{N}} \\
Also,\ \nabla_\nu \nabla_\mu \xi^\alpha &= R^\alpha{}_{\mu\nu\beta}\xi^\beta \\
So,\ \nabla_\nu \nabla_\mu \xi_\alpha &= R_{\alpha\mu\nu\beta}\xi^\beta \\
Now,\ \xi^\lambda \nabla_\lambda \kappa^2 &= \xi^\lambda \nabla_\lambda(-\frac{1}{2}(\nabla^\alpha \xi^\beta)(\nabla_\alpha \xi_\beta)) \\
\Rightarrow\ 2\kappa \partial_\lambda \kappa \xi^\lambda &= -(\nabla^\alpha \xi^\beta)\xi^\lambda \nabla_\lambda(\nabla_\alpha \xi_\beta) \\
&= -(\nabla^\mu \xi^\alpha)\xi^\nu \nabla_\nu(\nabla_\mu \xi_\alpha) \\
&= -(\nabla^\mu \xi^\alpha)\xi^\nu R_{\alpha\mu\nu\beta}\xi^\beta \\
&= -(\nabla^\mu \xi^\alpha)R_{\alpha\mu\nu\beta}\xi^\nu \xi^\beta \qquad (6.26) \\
&= -(\nabla^\mu \xi^\alpha)R_{\alpha\mu\beta\nu}\xi^\beta \xi^\nu \\
&= (\nabla^\mu \xi^\alpha)R_{\alpha\mu\nu\beta}\xi^\nu \xi^\beta \ (by\ Eq^n\ 2.20) \\
&= 0 \qquad (6.27)
\end{aligned}$$

Since, $\kappa \neq 0$; so, eq^n 6.27 implies :-

$$\partial_\lambda \kappa \xi^\lambda = 0 \tag{6.28}$$

$$\Rightarrow \xi \cdot \partial \kappa = 0 \tag{6.29}$$

$\therefore$ κ is constant along horizon's each null generator.

Now in transverse direction we have from eq^n 6.26 :-

$$2\kappa \partial_\lambda \kappa e^\lambda_A = -(\nabla^\mu \xi^\alpha) R_{\alpha\mu\nu\beta} e^\nu_A \xi^\beta \tag{6.30}$$

Now suppose, the black hole is geodesically complete. Then, it contains a Bifurcation 2-sphere; on which $\xi^\alpha = 0$. So, $\partial_\lambda \kappa e^\lambda_A = 0$ on the Bifurcation 2-sphere. Now,

$$\begin{aligned}
\xi^\mu \nabla_\mu (\partial_\lambda \kappa e^\lambda_A) &= \xi^\mu \partial_\mu (\partial_\lambda \kappa e^\lambda_A) \\
&= \left(\frac{\partial x^\mu}{\partial v}\right) \frac{\partial}{\partial x^\mu} \left(\frac{\partial \kappa}{\partial x^\lambda} \left(\frac{\partial x^\lambda}{\partial \theta^A}\right)\right) = \frac{\partial}{\partial v} \left(\frac{\partial \kappa}{\partial x^\lambda} \left(\frac{\partial x^\lambda}{\partial \theta^A}\right)\right) \\
&= \frac{\partial}{\partial v} \left(\frac{\partial \kappa}{\partial \theta^A}\right) = \frac{\partial^2 \kappa}{\partial v \partial \theta^A} = \frac{\partial^2 \kappa}{\partial \theta^A \partial v}
\end{aligned}$$

But $\frac{\partial \kappa}{\partial v} = 0$ as κ is constant horizon's null generators parametrized by v. So,

$$\xi^\mu \nabla_\mu (\partial_\lambda \kappa e^\lambda_A) = 0 \tag{6.31}$$

$\therefore$ $\kappa_{,\lambda} e^\lambda_A$ is constant on horizon's each null generator. This means the variation of κ along θ^A is constant along v (on null generators). Now, pick $v = c_1$(constant). Then, on this variation of κ along θ^A is constant. Now, at $v = c_2$(constant); again variation of κ along θ^A is constant but the value of κ here can be different than that of the value at $v = c_1$. But, this wouldn't be different as it has been already proved that κ is constant along v (on null generators). So, we see that $\kappa_{,\lambda} e^\lambda_A$ is constant on v = constant cross-sections of the event horizon. Now since at $(v, \theta^A) = (0, 0)$, i.e.; on the Bifurcation 2-sphere $\kappa_{,\lambda} e^\lambda_A = 0$; we conclude that $\forall$ v $\kappa_{,\lambda} e^\lambda_A = 0$, i.e.; it doesn't vary along v. Hence, finally from the above discussion we conclude that $\kappa_{,\lambda} e^\lambda_A = 0$ on v = constant cross-sections of the event horizon. Thus, κ doesn't vary from generator to generator. Hence, for a geodesically complete black hole; κ is uniform over the entire event horizon.

Now consider if the black hole is not geodesically complete. Then, say there are two black holes A and B, out of which A is geodesically complete and B is geodesically incomplete but both are identical in their space-times $\forall$ $v > 0$ (in future). Then, B is stationary only for $v > 0$ and A is stationary $\forall$ $v \epsilon \Re$. Now since $\kappa_{,\lambda} e^\lambda_A = 0$ on v = constant cross-sections of the event horizon of A; and furthermore, A and B are identical in their spacetimes $\forall$ $v > 0$ so, we conclude that $\forall$ $v > 0$; $\kappa_{,\lambda} e^\lambda_B = 0$ on v = constant cross-sections of the event horizon of B too.

This shows that for a stationary black hole the surface gravity κ is indeed uniform over the entire event horizon. **Proved**.

6.3.1 The Hawking Temperature

Sir *Hawking*'s discovery that quantum processes give rise to a thermal flux of particles from black holes implies they do indeed behave as thermodynamic systems. Black holes have a well-defined temperature, which as a matter of fact is proportional to the black hole's surface gravity κ given as :-

$$T_H = \frac{\kappa}{2\pi} \tag{6.32}$$

The Zeroth Law of Black Hole Mechanics can be seen therefore as a special case of the corresponding Law of Thermodynamics, which states that a system in thermal equilibrium has a uniform temperature.

6.4 Surface Gravity Calculations

In this section we present the surface gravity calculations of the 4−dimensional stationary black holes considered earlier.

6.4.1 Schwarzschild and Reissner-Nordström Metrics

Both the Schwarzschild and Reissner-Nordström metrics can be written in the form :-

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{6.33}$$

where $f(r)$ is a smooth real valued function of r. So, we have :-

$$g_{tt} = -f,\ g_{rr} = f^{-1},\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 \sin^2\theta \tag{6.34}$$

$$g^{tt} = -\frac{1}{f},\ g^{rr} = f\ g^{\theta\theta}, = \frac{1}{r^2},\ g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta} \tag{6.35}$$

Now consider the *Killing's Equation* :-

$$\nabla_{[\alpha}\xi_{\beta]} = 0$$

$$\Rightarrow\ \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = 2\Gamma^\lambda_{\alpha\beta}\xi_\lambda \tag{6.36}$$

Now let us compute the non-vanishing components of the *Affine Connection* :-

$$\Gamma^r_{tt} = \frac{ff^{'}}{2},\ \Gamma^r_{rr} = -\frac{f^{'}}{2f},\ \Gamma^r_{\theta\theta} = -fr \tag{6.37}$$

$$\Gamma^r_{\phi\phi} = -fr\sin^2\theta,\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{6.38}$$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{f^{'}}{2f},\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r},\ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{6.39}$$

Now observe since $g_{\mu\nu}$ is independent of t we have a straightforward *killing vector field* as :-

$$\xi = \frac{\partial}{\partial t}$$

$$\xi_t = 1$$

Now, putting $\xi_r = 0$, and intuitively setting up the remaining *Killing's Equation* we get :-

$$\frac{\partial \xi_\theta}{\partial\theta} = 0 \tag{6.40}$$

$$\frac{\partial \xi_\phi}{\partial\phi} = -\sin\theta\cos\theta\xi_\theta \tag{6.41}$$

$$\frac{\partial \xi_\phi}{\partial\theta} + \frac{\partial \xi_\theta}{\partial\phi} = 2\cot\theta\xi_\phi \tag{6.42}$$

The above $eq^n s$ are similar to the ones we set up for the *killing vector fields* of S^2. Hence, linearly independent *killing vector fields* of given spacetime are :-

$$\xi_1 = \frac{\partial}{\partial t} \tag{6.43}$$

$$\xi_2 = \frac{\partial}{\partial\phi} \tag{6.44}$$

$$\xi_3 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \tag{6.45}$$

$$\xi_4 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \tag{6.46}$$

For calculation of the surface gravity of the black hole consider the metric in *ingoing Eddington-Finkelstein* coordinates (v, r, θ, ϕ) :-

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6.47}$$

where,

$$v = t + r^* \tag{6.48}$$

$$r^* = \int \frac{dr}{f(r)} \tag{6.49}$$

Furthermore, roots of $f(r) = 0$ will be $r = r_{\pm}$ in case of Reissner-Nordström black hole and $r = r_H$ in case of Schwarzschild black hole.

So, for $v = constant$ we have *ingoing* rays and v is known as the advance time coordinate. Furthermore, the metric in this form clearly suggests that root of $f(r) = 0$ is just a coordinate singularity as the metric is well behaved there. Now for computation of κ just consider :-

$$ds^2 = -fdv^2 + 2dvdr \tag{6.50}$$

Now,

$$g_{\mu\nu} = \begin{pmatrix} -f & 1 \\ 1 & 0 \end{pmatrix};\ g^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & f \end{pmatrix} \tag{6.51}$$

Now,

$$\nabla_r \xi_r = 0 \ \Rightarrow\ \partial_r \xi_r = \Gamma^{\lambda}_{rr}\xi_\lambda = 0 \tag{6.52}$$

$$\Rightarrow\ \xi_r = g(v)$$

$$\partial_v \xi_v = \Gamma^{\lambda}_{vv} = \frac{f'}{2}\xi_v + \frac{ff'}{2}\xi_r$$

$$So,\ \frac{\partial \xi_v}{\partial v} = \frac{f'}{2}\xi_v + \frac{ff'}{2}g(v) \tag{6.53}$$

$$\frac{\partial \xi_r}{\partial v} + \frac{\partial \xi_v}{\partial r} = 2\Gamma^{\lambda}_{rv}\xi_\lambda = -f'\xi_r = -f'g(v)$$

$$\Rightarrow\ \frac{\partial \xi_v}{\partial r} = -f'g(v) - \frac{\partial g(v)}{\partial v} \tag{6.54}$$

$$Upon\ Integrating\ w.r.t.\ r;\ \xi_v = -gf - \frac{\partial g}{\partial v}r + h(v) \tag{6.55}$$

$$So,\ \frac{\partial \xi_v}{\partial v} = -fg' - rg'' + h' \tag{6.56}$$

$$= \frac{f'}{2}[h - rg' - gf] + \frac{ff'}{2}g\ (by\ eq^n\ 6.53)$$

$$= \frac{hf'}{2} - \frac{rf'g'}{2} \tag{6.57}$$

$$From\ eq^n\ 6.56\ and\ eq^n\ 6.57;\ -fg' - rg'' + h' = \frac{hf'}{2} - \frac{rf'g'}{2} \tag{6.58}$$

Let in eq^n 6.58; $h = 0$ and $g = c_1(constant)$. This choice satisfies eq^n 6.58. So, we get taking $c_1 = 1$ for linear independence :-

$$\xi_v = -f,\ \xi^v = 1 \tag{6.59}$$

$$\xi_r = 1,\ \xi^r = 0 \tag{6.60}$$

$$\Rightarrow\ \xi^\mu \xi_\mu = \xi^v \xi_v = -f \tag{6.61}$$

$$Furthermore,\ |\xi^\mu \xi_\mu|_{r=r_\pm/r_H} = 0 \tag{6.62}$$

So, indeed $r = r_{\pm}/r_H$ are/is killing horizon(s) to the *killing vector field* :-

$$\xi = \frac{\partial}{\partial v} \tag{6.63}$$

Now,

$$g^{\mu\nu}(\nabla_\mu \xi_\nu)^2 = g^{rr}(\nabla_r \xi_r) + g^{rv}(\nabla_r \xi_v) + g^{vr}(\nabla_v \xi_r) = g^{rv}(\nabla_r \xi_v) + g^{vr}(\nabla_v \xi_r)$$

$$But,\ \nabla_r \xi_v = -\nabla_v \xi_r$$

$$So,\ (\nabla_r \xi_v)^2 = (\nabla_v \xi_r)^2$$

$$Hence,\ g^{\mu\nu}(\nabla_\mu \xi_\nu)^2 = 2(\nabla_r \xi_v)^2$$

$$Now,\ \nabla_r \xi_v = \partial_r \xi_v - \Gamma^r_{rv}\xi_r = -f' + \frac{f'}{2} = -\frac{f'}{2}$$

$$Finally,\ (\nabla_r \xi_v)^2 = \left(\frac{f'}{2}\right)^2$$

We know :-

$$\begin{aligned}\kappa^2 &= -\frac{1}{2}\left|g^{\mu\nu}(\nabla_\mu\xi_\nu)^2\right|_{r=r_\pm/r_H}\\ &= -\frac{1}{2}2\left|(\nabla_r\xi_v)^2\right|_{r=r_\pm/r_H}\\ &= -\left|\left(\frac{f'}{2}\right)^2\right|_{r=r_\pm/r_H}\end{aligned}$$

Choosing the +ve sign for κ we arrive (denoting RN for Reissner-Nordström) at :-

$$\kappa_{RN_+} = \frac{f'(r_+)}{2} = \frac{(M^2-Q^2)^{\frac{1}{2}}}{(M+(M^2-Q^2)^{\frac{1}{2}})^2} \tag{6.64}$$

$$\kappa_{RN_-} = \frac{f'(r_+)}{2} = \frac{(M^2-Q^2)^{\frac{1}{2}}}{(M-(M^2-Q^2)^{\frac{1}{2}})^2} \tag{6.65}$$

$$\kappa_{Schwarzschild} = \frac{f'(r_H)}{2} = \frac{1}{4M} \tag{6.66}$$

Hence we also obtain :-

$$T_{H_{RN}} = \frac{\kappa_{RN_+}}{2\pi} = \frac{(M^2-Q^2)^{\frac{1}{2}}}{2\pi(M+(M^2-Q^2)^{\frac{1}{2}})^2}\ (as\ r=r_+\ is\ the\ event\ horizon) \tag{6.67}$$

$$T_{H_{Schwarzschild}} = \frac{\kappa_{Schwarzschild}}{2\pi} = \frac{1}{8\pi M} \tag{6.68}$$

6.4.2 Kerr and Kerr-Newman Metrics

Here we present the computation of surface gravity of the Kerr black hole. The surface gravity of the Kerr-Newman black hole has exactly the same computation as that of the Kerr only with very slight modifications, so we will directly present the result for the Kerr-Newman case.

Now, the null hypersurfaces are $r = r_\pm$ and inspection of the Kerr metric suggests two obvious linearly independent *killing vector fields* :-

$$\xi_1 = \frac{\partial}{\partial t} \tag{6.69}$$

$$\xi_2 = \frac{\partial}{\partial \phi} \tag{6.70}$$

We take their linear combination as :-

$$\xi_\pm = \frac{\partial}{\partial t} + \Omega_{H_\pm}\frac{\partial}{\partial\phi}$$

Now, we would like to make $r = r_\pm$ as *killing horizons* for $\xi_\pm$. So, we seek the value of $\Omega_{H_\pm}$ such that $|\xi^\mu\xi_\mu|_{r=r_\pm} = 0$. So,

$$\begin{aligned}\xi^\mu\xi_\mu &= \xi^t\xi_t + \xi^\phi\xi_\phi\\ &= g_{tt} + g_{t\phi}\Omega_H + \Omega_H(g_{\phi\phi}+g_{\phi t})\\ &= {\Omega_H}^2 g_{\phi\phi} + 2\Omega_H g_{\phi t} + g_{tt}\\ &= {\Omega_H}^2\left[\frac{\Sigma\sin^2\theta}{\rho^2}\right] - 2\Omega_H\left(\frac{\omega\Sigma\sin^2\theta}{\rho^2}\right) + \frac{\omega\Sigma\sin^2\theta}{\rho^2}\omega^2 - \frac{\rho^2\Delta}{\Sigma}\\ &= \frac{\Sigma\sin^2\theta}{\rho^2}[\Omega_H-\omega]^2 - \frac{\rho^2\Delta}{\Sigma}\end{aligned}$$

Now at $r = r_\pm$, $\Delta = 0$. So,

$$\begin{aligned}|\xi^\mu\xi_\mu|_{r=r_\pm} &= \frac{\Sigma_\pm\sin^2\theta}{{\rho_\pm}^2}[\Omega_{H_\pm}-\omega_\pm]^2\\ &= \frac{({r_\pm}^2+a^2)^2\sin^2\theta}{{r_\pm}^2+a^2\cos^2\theta}[\Omega_{H_\pm}-\omega_\pm]^2 \end{aligned}\tag{6.71}$$

Now from eq^n 6.71, $|\xi^\mu \xi_\mu|_{r=r_\pm} = 0$ holds only if $\Omega_{H_\pm} = \omega_\pm = \frac{a}{{r_\pm}^2+a^2}$. Now, dimensionally; $[\Omega_H] = [\omega]$ and hence, on $r = r_+$ (the event horizon); Ω_{H_+} is termed as the angular velocity of the black hole. So, we get the desired *killing vector field* as :-

$$\xi_\pm = \frac{\partial}{\partial t} + \Omega_{H_\pm} \frac{\partial}{\partial \phi} = \frac{\partial}{\partial t} + \frac{a}{{r_\pm}^2 + a^2} \frac{\partial}{\partial \phi} \tag{6.72}$$

We know that the normal vector field is, $l_\mu = \frac{\partial r}{\partial x^\mu}$. Now, we know since $\xi_\pm$ is normal to $r = r_\pm$; we have :-

$$\xi_\mu = c_1 \frac{\partial r}{\partial x^\mu} \tag{6.73}$$

$$Furthermore,\ \xi^\mu \xi_\mu = g^{\mu\nu} \xi_\mu \xi_\nu = g^{\mu\nu} {c_1}^2 \partial_\mu r \partial_\nu r = {c_1}^2 g^{rr}$$

$$\Rightarrow\ {c_1}^2 = \frac{\xi^\mu \xi_\mu}{g^{rr}} = \frac{\rho^2}{\Delta} \frac{\Sigma \sin^2\theta}{\rho^2} [\Omega_H - \omega]^2 - \frac{\rho^2 \Delta}{\Sigma}$$

$$\Rightarrow\ \left|{c_1}^2\right|_{r=r_\pm} = \Sigma_\pm \sin^2\theta \lim_{r \to r_\pm} \frac{\Omega_H - \omega}{\Delta} - \frac{{\rho_\pm}^4}{\Sigma_\pm}$$

$$= \Sigma_\pm \sin^2\theta \left| 2(\Omega_H - \omega) \frac{\partial \omega}{\partial r} \right|_{r=r_\pm} \frac{1}{2(r_\pm - M)} - \frac{{\rho_\pm}^4}{\Sigma_\pm}$$

$$= \frac{{\rho_\pm}^4}{\Sigma_\pm}$$

$$|c_1|_{r=r_\pm} = \frac{{\rho_\pm}^2}{{\Sigma_\pm}^{\frac{1}{2}}} \tag{6.74}$$

We know, $|\nabla_\mu(-\xi^\alpha \xi_\alpha)|_{r=r_\pm} = |2\kappa \xi_\mu|_{r=r_\pm}$. Furthermore,

$$\frac{{\rho_\pm}^4}{\Sigma_\pm} = \frac{{\rho_\pm}^2}{\Sigma_\pm} 2(r_\pm - M) \partial_\mu r = 2\kappa_\pm \frac{{\rho_\pm}^2}{{\Sigma_\pm}^{\frac{1}{2}}} \partial_\mu r$$

Hence;

$$\kappa_{K_\pm} = \frac{r_\pm - M}{{r_\pm}^2 + a^2} \tag{6.75}$$

So, we finally obtain :-

$$T_{H_K} = \frac{\kappa_{K_+}}{2\pi} = \frac{r_+ - M}{2\pi({r_+}^2 + a^2)} \tag{6.76}$$

where K denotes Kerr.
Now, in the case of Kerr-Newman black hole the exact result holds with $r_\pm = M \pm (M^2 - a^2 - Q^2)^{\frac{1}{2}}$. So,

$$\kappa_{KN_\pm} = \frac{r_\pm - M}{{r_\pm}^2 + a^2} \tag{6.77}$$

So, we finally obtain :-

$$T_{H_{KN}} = \frac{\kappa_{KN_+}}{2\pi} = \frac{r_+ - M}{2\pi({r_+}^2 + a^2)} \tag{6.78}$$

where KN denotes Kerr-Newman.

6.4.3 Extreme Black Holes

In the extreme cases of Reissner-Nordström, Kerr and Kerr-Newman black holes one can see that they all yield identically the same surface gravity, i.e.;

$$\kappa_{ex} = 0\ \Rightarrow\ T_{H_{ex}} = 0 \tag{6.79}$$

6.5 Black Holes Uniqueness Theorems

In this section we qualitatively present some of the black hole uniqueness theorems in a 4−dimensional spacetime manifold. In the absence of any matter in their exterior, stationary black holes admit an extremely simple description.

6.5.1 Static Black Holes

If the black hole is static, then it must be spherically symmetric and it can only be described by the Schwarzschild solution. It implies that in the absence of angular momentum, complete gravitational collapse must result in a Schwarzschild black hole. This might seem puzzling, because the statement is true irrespective of the initial shape of the progenitor, which might have been strongly nonspherical. If the black hole is static and carries an electric charge, then it must be a Reissner-Nordström black hole

6.5.2 Axially Symmetric Black Holes

If the black hole is axially symmetric, then it must be a Kerr black hole. Furthermore, if it also carries an electric charge then it must be a Kerr-Newman black hole.

6.5.3 Prelude to the *No Hair Theorem*

Thus, we see that a black hole in isolation can be characterized, uniquely and completely, by just three parameters: its mass, angular momentum, and charge. No other parameter is required, and this remarkable property is at the origin of the Black Holes' *No Hair theorem.*

These above black hole uniqueness theorems are one of the main reasons why the event horizons of the black holes in $4-$dimensions always have spherical topologies, i.e.; of S^2.

Finally, we end this section by quoting S.Chandrasekhar :- *"Black holes are macroscopic objects with masses varying from a few solar masses to millions of solar masses. To the extent that they may be considered as stationary and isolated, to that extent, they are all, every single one of them, described exactly by the Kerr solution. This is the only instance we have of an exact description of a macroscopic object. Macroscopic objects, as we see them all around us, are governed by a variety of forces, derived from a variety of approximations to a variety of physical theories. In contrast, the only elements in the construction of black holes are our basic concepts of space and time. They are, thus, almost by definition, the most perfect macroscopic objects there are in the universe. And since the general theory of relativity provides a single unique two-parameter family of solutions for their descriptions, they are the simplest objects as well."*

6.6 $D > 4$ Black Holes

In $d > 4$ dimensions, there exists multitudes of solutions to *Einstein's Field Equations* whose event horizons have non-spherical topologies. Also their characterization requires more than three parameters. So, the uniqueness theorems for the solutions no longer holds generally for $d > 4$ dimensions.

6.6.1 Myers-Perry Black Holes

Myers-Perry black holes are the generalization of Kerr black hole to higher dimensional spacetime. In this section we would compute their surface gravities. Also we would denote $\mu = 2M$ and $a_i = \frac{J_i}{M}$ where M is the mass and J_is are angular momenta of the black hole in specific directions.

Flat metric in Higher Odd dimensions

Let $d = 2n + 1$ where, $2 \leq n \epsilon \mathbb{N}$.

Now,

$$ds^2 = -dt^2 + \sum_{i=1}^{n} dx_i{}^2 + dy_i{}^2 \tag{6.80}$$

Now, define for $\mu_i \epsilon \Re$, where $i \epsilon \{1, 2, \cdots\cdots, n\}$:-

$$x_i = r\mu_i \cos\phi_i, \ \ y_i = r\mu_i \sin\phi_i \tag{6.81}$$

Now using;

$$dx_i = \mu_i \cos\phi_i dr + r\cos\phi_i d\mu_i - r\mu_i \sin\phi_i d\phi_i$$
$$dy_i = \mu_i \sin\phi_i dr + r\sin\phi_i d\mu_i + r\mu_i \cos\phi_i d\phi_i$$

Furthermore, using :-

$$\sum_{i=1}^{n} x_i^2 + y_i^2 = r^2(\mu_i^2 + \mu_i^2 \sin 2\phi_i) = r^2$$
$$\Rightarrow \sum_{i=1}^{n} \mu_i^2(1 + \sin 2\phi_i) = 1$$
$$\Rightarrow \sum_{i=1}^{n} \mu_i^2 = 1 \ (as, \ \int \sin 2\phi_i d\phi_i \to 0)$$

So, we finally obtain :-

$$ds^2 = -dt^2 + dr^2 + r^2 \sum_{i=1}^{n} (d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{6.82}$$

$$Constraint; \ \sum_{i=1}^{n} \mu_i^2 = 1 \tag{6.83}$$

Flat metric in Higher Even dimensions

Let $d = 2n + 2$ where, $1 \leq n\epsilon\mathbb{N}$.

Now,

$$ds^2 = -dt^2 + dz^2 + \sum_{i=1}^{n} dx_i{}^2 + dy_i{}^2 \tag{6.84}$$

Now, define for $\mu_i, \alpha\epsilon\Re$, where $i\epsilon\{1, 2, \cdots\cdots, n\}$:-

$$x_i = r\mu_i \cos\phi_i, \ y_i = r\mu_i \sin\phi_i, \ z = r\alpha \tag{6.85}$$

As before one can exactly compute the same way to obtain :-

$$ds^2 = -dt^2 + dr^2 + r^2 d\alpha^2 + r^2 \sum_{i=1}^{n} (d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{6.86}$$

$$Constraint; \ \alpha^2 + \sum_{i=1}^{n} \mu_i^2 = 1 \tag{6.87}$$

Myers-Perry metric in $d = 2n + 1$ dimension

Here, $d = 2n + 1$ and $n \geq 2$

$$ds^2 = -dt^2 + \frac{\mu r^2}{\Pi F}\left(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 + \sum_{i=1}^{n} (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) \tag{6.88}$$

$$Constraint, \ \sum_{i=1}^{n} \mu_i^2 = 1 \tag{6.89}$$

where,

$$F = 1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \tag{6.90}$$

$$\Pi = \prod_{i=1} (r^2 + a_i^2) \tag{6.91}$$

Myers-Perry metric in $d = 2n+2$ dimension

Here, $d = 2n+2$ and $n \geq 1$

$$ds^2 = -dt^2 + \frac{\mu r}{\Pi F}\left(dt + \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r}dr^2 + \sum_{i=1}^{n}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 d\alpha^2 \tag{6.92}$$

$$Constraint,\ \alpha^2 + \sum_{i=1}^{n} \mu_i^2 = 1 \tag{6.93}$$

where,

$$F = 1 - \sum_{i=1}^{n} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \tag{6.94}$$

$$\Pi = \prod_{i=1} (r^2 + a_i^2) \tag{6.95}$$

Family of *hypersurfaces*

For $d = 4$ case the even dimensional Myers-Perry metric reduces to the familiar Kerr metric. So, motivated by $d = 4$; we take $r = constant$ as the family of *hypersurfaces* containing the horizons. Also, from the above Myers-Perry metrics we see that $const(t, r)$ have indeed spherical topology, i.e.; S^{d-2} topology. Furthermore, like before; roots of $g^{rr} = 0$ describe the horizons. So,

$$Odd\ case\ :-\ \Pi - \mu r^2 = 0\ (describe\ horizons) \tag{6.96}$$

$$Even\ case\ :-\ \Pi - \mu r = 0\ (describe\ horizons) \tag{6.97}$$

Also observe that intuitively we can deduce that since at the roots of the above $eq^n s$ the $r-$coordinate changes sign; hence, these definitely account for the horizons. Let us denote the roots (only the event horizons) in both the cases by $r = r_H$.

Normal Vector Fields calculation

In both cases for $\tilde{f} \epsilon \mathcal{F}$:-

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu} \frac{\partial r}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\ &= \tilde{f} g^{rr} \frac{\partial}{\partial r} \\ Now,\ l^r l_r &= \tilde{f}^2 g^{rr} \\ So,\ \left| l^\mu l_\mu \right|_{r=r_H} &= 0 \end{aligned}$$

So, this shows that in both the cases $r = r_H$ is a indeed null *hypersurface*.

***Killing Vector Field* calculation for odd dimensions**

Observing the Myers-Perry metric for the odd dimensions we observe that $g_{\mu\nu}$ is independent of (t, ϕ_i). Hence motivated by the Kerr case we demand the *killing vector field* to be :-

$$\xi = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial \phi_i}$$

Now as previously we seek to find β_i such that $\left| \xi^\mu \xi_\mu \right|_{r=r_H} = 0$.

Now,

$$\begin{aligned} \xi^\mu \xi_\mu = g_{\mu\nu}\xi^\mu \xi^\nu &= g_{tt}\xi^t \xi^t + 2g_{\phi_i \phi_j}\xi^{\phi_i}\xi^{\phi_j} + g_{\phi_i \phi_i}\xi^{\phi_i}\xi^{\phi_i} + 2g_{t\phi_i}\xi^t \xi^{\phi_i} \\ &= \left(\frac{\mu r^2}{\Pi F} - 1\right) + \left[\sum_{i=1}^{n} \frac{\mu r^2}{\Pi F} a_i^2 \mu_i^4 + r^2 \mu_i^4 + a_i^2 \mu_i^2\right] + \left(\frac{2\mu r^2}{\Pi F}\sum_{i=1}^{n} a_i \mu_i^2\right)\beta_i + \left(\frac{2\mu r^2}{\Pi F}\sum_{i=1}^{n-1}\sum_{j>i}^{n} a_i a_j \mu_i^2 \mu_j^2\right)\beta_i \beta_j \end{aligned}$$

Now, $|\xi^\mu \xi_\mu|_{r=r_H} = 0$ gives :-

$$\left(\sum_{i=1}^{n} \frac{a_i^2\mu_i^2}{r_H^2 + a_i^2}\right) + \sum_{i=1}^{n} a_i^2\mu_i^4\beta_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{r_H^4\mu_i^2\mu_j^2}{r_H^2 + a_j^2}\beta_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{r_H^2\mu_i^2 a_i^2\mu_j^2}{r_H^2 + a_j^2}\beta_i^2 + 2\sum_{i=1}^{n} a_i\mu_i^2\beta_i + \sum_{i=1}^{n-1}\sum_{j>i}^{n} a_i a_j \mu_i^2\mu_j^2\beta_i\beta_j = 0$$

Now, using the constraint relation, $\sum\limits_{i=1}^{n} \mu_i^2 = 1$; we see that :-

$$\mu_1^2 = 1 - \sum_{i=2}^{n} \mu_i^2 \tag{6.98}$$

Using this in above equation we see that the constant terms obtained on putting $i, j = 1$ is :-

$$\frac{a_1^2}{r_H^2 + a_1^2} + 2a_1\beta_1 + a_1^2\beta_1^2 + \frac{r_H^2}{r_H^2 + a_1^2}\beta_1^2 + \frac{a_1^2 r_H^2}{r_H^2 + a_1^2}\beta_1^2 \tag{6.99}$$

But since $|\xi^\mu \xi_\mu|_{r=r_H} = 0$ this implies that expression 6.99 must vanish as it just contains the constant term of $|\xi^\mu \xi_\mu|_{r=r_H}$. Hence;

$$\frac{a_1^2}{r_H^2 + a_1^2} + 2a_1\beta_1 + a_1^2\beta_1^2 + \frac{r_H^2}{r_H^2 + a_1^2}\beta_1^2 + \frac{a_1^2 r_H^2}{r_H^2 + a_1^2}\beta_1^2 = 0$$
$$\beta_1^2(2a_1^2 r_H^2 + a_1^4 + r_H^4) + 2a_1(r_H^2 + a_1^2)\beta_1 + a_1^2 = 0$$
$$\Rightarrow \ \beta_1^2(a_1^2 + r_H^2)^2 + 2a_1(r_H^2 + a_1^2)\beta_1 + a_1^2 = 0$$
$$\Rightarrow \ \beta_1 = -\frac{a_1}{a_1^2 + r_H^2} \tag{6.100}$$
$$Similarly,\ for\ other\ \beta_i's\ we\ get;\ \beta_i = -\frac{a_i}{a_i^2 + r_H^2} \tag{6.101}$$

So, we finally obtain :-

$$\xi_{odd} = \frac{\partial}{\partial t} - \frac{a_i}{a_i^2 + r_H^2}\frac{\partial}{\partial \phi_i} \tag{6.102}$$

This makes $r = r_H$ a *killing horizon* for ξ_{odd}.

Killing Vector Field calculation for even dimensions

Here consider the constraint relation, $\alpha^2 + \sum\limits_{i=1}^{n} \mu_i^2 = 1$. So,

$$d\alpha = -\frac{1}{\alpha}\sum_{i=1}^{n} \mu_i d\mu_i$$
$$d\alpha^2 = \frac{1}{\alpha^2}\left(\sum_{i=1}^{n} \mu_i d\mu_i\right)^2$$

So, as $z = r\alpha$:-

$$dz^2 = r^2 d\alpha^2 = \frac{r^2}{\alpha^2}\left(\sum_{i=1}^{n} \mu_i d\mu_i\right)^2 \tag{6.103}$$

From eq^n 6.103 we see that dz^2 doesn't contain constant term since there is always an $\frac{1}{\alpha^2}$ term present in all of its terms $\forall\ i\{1, 2, \cdots\cdots, n\}$. Hence the *killing vector field* doesn't differ from that of the odd case :-

$$\xi_{even} = \frac{\partial}{\partial t} - \frac{a_i}{a_i^2 + r_H^2}\frac{\partial}{\partial \phi_i} \tag{6.104}$$

This makes $r = r_H$ a *killing horizon* for ξ_{even}.

Surface gravity in odd dimensions

Consider, $\xi^\mu \nabla_\mu \xi^\alpha = \kappa \xi^\alpha$; this with $\xi_\alpha = c_1 \partial_\alpha r (c_1 \; is \; constant \epsilon \Re)$ gives :-

$$\xi^r \nabla_r \xi^r = \xi^r \partial_r \xi^r + \xi^r \Gamma^r_{r\alpha} \xi^\alpha = \kappa \xi^r \; (as \; \xi \; has \; only \; r - coordinate)$$

$$\Rightarrow \; \kappa = c_1 [\partial_r g^{rr} + \frac{g^{rr2}}{2} \partial_r g_{rr}] \; (as \; \Gamma^r_{rr} \; is \; the \; only \; non-vanishing \; in \; given \; case)$$

$$= c_1 [\partial_r g^{rr} + \frac{g^{rr2}}{2} \partial_r g^{rr-1}]$$

$$= c_1 [\partial_r g^{rr} - \frac{g^{rr2}}{2} \frac{1}{g^{rr2}} \partial_r g^{rr}$$

$$= \frac{c_1}{2} \partial_r g^{rr}$$

So,

$$\kappa = \frac{c_1}{2} \partial_r g^{rr} \tag{6.105}$$

Now we know that, $c_1^2 = \lim\limits_{r \to r_H} \frac{\xi^\mu \xi_\mu}{g^{rr}} = \lim\limits_{r \to r_H} \frac{\partial_r (\xi^\mu \xi_\mu)}{\partial_r (g^{rr})}$. Now, let us deduce c_1^2 without actual computation (since its too long and tedious) but by using mathematical logic as follows.

Consider,

$$|\partial_r g^{rr}|_{r_H} = \left| \frac{\partial_r \Pi - 2\mu r}{\mu r^2 F} \right|_{r_H} \neq 0$$

So, limit exists. As limit exists and $|\partial_r g^{rr}|_{r_H} \neq 0$ and furthermore, since the metric doesn't describe a extreme black hole so; $|\partial_r g(\xi^\mu \xi_\mu)|_{r_H} \neq 0$. Given metric doesn't describe a extreme black hole as other real roots to $\Pi - \mu r^2 = 0$ exists other than $r = r_H$. So, in general it is not the extreme case. Now from eq^n 6.105 we see that :-

$$\kappa = \frac{c_1}{2} \frac{\partial_r \Pi - 2\mu r}{\mu r^2 F}$$

Now at $r = r_H$; $\frac{\partial_r \Pi - 2\mu r}{\mu r^2}$ is constant. But, since; $F = F(r, \mu_i)$ it is not a constant at $r = r_H$. Now, since by the zeroth law we have that κ must be a constant throughout the entire event horizon we observe that at $r = r_H$:-

$$c_1 = \gamma F \tag{6.106}$$

for some constant $\gamma \epsilon \Re$. c_1 can't be a function of r as limit $r \to r_H$ is taken to get c_1. This gives :-

$$\kappa_{odd} = \left| \gamma \frac{\partial_r \Pi - 2\mu r}{2\mu r^2} \right|_{r = r_H} \tag{6.107}$$

Surface gravity in even dimensions

Exactly following similar logic and calculation we get with just g^{rr} modified in the even dimensions :-

$$\kappa_{even} = \left| \gamma \frac{\partial_r \Pi - \mu}{2\mu r} \right|_{r = r_H} \tag{6.108}$$

Deducing the value of γ from κ_{Kerr}

For Kerr, $d = 4$ and $n = 1$. So,

$$\Pi = r^2 + a_1^2; \; \partial_r \Pi = 2r$$

$$Also, \; r^2 + a_1^2 - \mu r = 0$$

$$\Rightarrow r = \frac{\mu \pm (\mu^2 - 4a_1^2)^{\frac{1}{2}}}{2}$$

Now,

$$\kappa_{Kerr} = \left| \gamma \frac{\partial_r \Pi - \mu}{2\mu r} \right|_{r = r_H}$$

$$= \gamma \frac{r_H - \frac{\mu}{2}}{r_H^2 + a_1^2}$$

Since, $\mu = 2M$; we get :-

$$\kappa_{Kerr} = \gamma \frac{r_H - M}{r_H^2 + a_1^2} \tag{6.109}$$

Comparing this with the previous obtained result we see that $\gamma = 1$.

Hence, we get for;
Odd d :-

$$\kappa_{odd} = \left| \frac{\partial_r \Pi - 2\mu r}{2\mu r^2} \right|_{r=r_H} \tag{6.110}$$

$$T_{H_{odd}} = \left| \frac{\partial_r \Pi - 2\mu r}{4\pi\mu r^2} \right|_{r=r_H} \tag{6.111}$$

Even d :-

$$\kappa_{even} = \left| \frac{\partial_r \Pi - \mu}{2\mu r} \right|_{r=r_H} \tag{6.112}$$

$$T_{H_{even}} = \left| \frac{\partial_r \Pi - \mu}{4\pi\mu r} \right|_{r=r_H} \tag{6.113}$$

So, Myers-Perry even metric is a higher dimensional generalization of the Kerr metric.

6.6.2 Higher dimensional Schwarzschild and RN cases

Higher dimensional Schwarzschild metric is of the form :-

$$-\left(1 - \frac{\mu}{r^{d-3}}\right) + \left(1 - \frac{\mu}{r^{d-3}}\right)^{-1} + r^2 d\Omega_{d-2}^2 \tag{6.114}$$

where $\mu = 2M$ and $d\Omega_{d-2}^2$ is the S^{d-2} metric. Clearly for $d = 4$ case we obtain the Schwarzschild metric. Observe that the above metric can be made into the form as that of eq^n 6.33 with only the metric of the sphere being changed with doesn't affect surface gravity calculations. So, the surface gravity would turn out to be the same as before in both higher dimensional Schwarzschild and RN cases.

Now, $r^{d-3} = const$ is the family of *hypersurfaces* containing horizons and $r^{d-3} = \mu$ describes the horizons. Now,

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu} \frac{\partial r}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\ &= \tilde{f} g^{rr} \frac{\partial}{\partial r} \\ Now,\ l^r l_r &= \tilde{f}^2 g^{rr} \\ So,\ |l^\mu l_\mu|_{r=r_H} &= 0 \end{aligned}$$

So, indeed roots of $r^{d-3} = \mu$ are null *hypersurfaces*.

Like the Schwarzschild case we can select the *killing vector field* to be :-

$$\xi = \frac{\partial}{\partial t} \tag{6.115}$$

$$So,\ \xi^\mu \xi_\mu = \xi^t \xi_t = g_{tt} = -\left(1 - \frac{\mu}{r^{d-3}}\right)$$

Now, $\xi_\alpha = c_1 \partial_\alpha r$; so,

$$\xi^\alpha \xi_\alpha = c_1^2 g^{rr} = c_1^2 \left(1 - \frac{\mu}{r^{d-3}}\right)$$

Hence, we see taking the +ve sign that $c_1 = 1$. So,

$$\xi_\alpha = \partial_\alpha r \tag{6.116}$$

Now, using $\nabla_\alpha(-\xi^\mu \xi_\mu) = 2\kappa\xi_\alpha$; we see :-

$$\partial_\alpha(f(r)) = 2\kappa\xi_\alpha \ (as \ \xi^\mu\xi_\mu = -f(r))$$
$$\Rightarrow \ f'(r)\partial_\alpha r = 2\kappa\partial_\alpha r$$

where $f(r) = \left(1 - \frac{\mu}{r^{d-3}}\right)$.

Hence;

$$\kappa = \left|\frac{f'(r)}{2}\right|_{r=r_H} \tag{6.117}$$
$$T_H = \left|\frac{f'(r)}{4\pi}\right|_{r=r_H} \tag{6.118}$$

6.7 Black Rings

Definition 6.7.1. Black rings are black holes in 5−dimensional spacetime with topology of their event horizons as $S^1 \otimes S^2$.

Note. A circular string has $\Re \otimes S^2$ topology and so intuitively string bent into a circle has $S^1 \otimes S^2$ topology.

Remark. The black ring due to the ring's own gravitation might try to self contract onto itself if the ring is static. Hence to counter balance the self gravity, stable black ring solutions must correspond to rotating black rings where the inward gravity is counterbalanced by the outward centrifugal force.

6.7.1 Neutral Black Ring

Consider flat metric in 4−dimensional space with coordinates (x^1, x^2, x^3, x^4) as :-

$$ds_4^2 = dr_1^2 + r_1^2 d\phi_2 + dr_2^2 + r_2^2 d\psi^2 \tag{6.119}$$

where $x^1 = r_1\cos\phi, x^2 = r_1\sin\phi, x^3 = r_2\cos\psi, x^4 = r_2\sin\psi$; done by grouping (x^1, x^2) and (x^3, x^4). Now, define new coordinates (x, y, ϕ, ψ) :-

$$r_1 = R\frac{(1-x^2)^{\frac{1}{2}}}{x-y}; \ r_2 = R\frac{(y^2-1)^{\frac{1}{2}}}{x-y} \tag{6.120}$$
$$with, \ -\infty \le y \le -1; \ -1 \le x \le 1$$

where R has dimensions of length, and for thin large rings it corresponds roughly to the radius of the ring circle.

In this coordinate system the metric takes the following form :-

$$ds_4^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)d\phi^2\right] \tag{6.121}$$

From the ranges of x, y values and observing the metric it can be seen that $y = -\infty$ indeed makes the metric truly singular and it therefore refers to the ring's source where the singularity is located. Asymptotic limit is recovered at $x \to y \to -1$.

Now, define new coordinates (r, θ, ϕ, ψ) :-

$$r = -\frac{R}{y}; \ \cos\theta = x \tag{6.122}$$
$$with, 0 \le r \le R; \ 0 \le \theta \le \pi \tag{6.123}$$

So, the metric becomes :-

$$ds_4^2 = \frac{1}{\left(1 + \frac{r\cos\theta}{R}\right)^2}\left[\left(1 - \frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{6.124}$$

Let $r = r_c$ be constant in the above metric, then the metric becomes :-

$$ds_4^2 = \frac{1}{\left(1 + \frac{r_c \cos\theta}{R}\right)^2} \left[\left(1 - \frac{r_c^2}{R^2}\right) R^2 d\psi^2 + r_c^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \tag{6.125}$$

Now we know S^1 metric is $ds^2 = r^2 d\psi^2$; and S^2 metric is $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$. So, observing the above metric one can clearly see that it is the metric for $S^1 \otimes S^2$ topology. This intuitively justifies that black rings have $S^1 \otimes S^2$ topology for their event horizons (constant r then time slicing, i.e; constant t). Also, notice that constant $r \Rightarrow$ constant y. This intuitively motivates us to choose $y = constant$ as the family of *hypersurfaces* containing the event horizon. Now, let us write down the metric for the neutral rotating black ring containing a rotating 2−sphere as :-

$$ds^2 = -\frac{F(y)}{F(x)} \left(dt - CR\frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \left[-\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right] \tag{6.126}$$

where $F(z) = 1 + \lambda z$; $G(z) = (1 - z^2)(1 + \nu z)$; $C = \left(\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}\right)^{\frac{1}{2}}$, with; $0 < \nu \leq \lambda < 1$. Now if $\lambda = \nu = 0$ then, we obtain the flat metric in the form as eq^n 6.121 .

To remove a conical singularity at $x = 1$ we have to relate λ with ν as :-

$$\lambda = \frac{2\nu}{1 + \nu^2} \tag{6.127}$$

$$\frac{1-\lambda}{1+\lambda} = \left(\frac{1-\nu}{1+\nu}\right)^2 \tag{6.128}$$

Now consider the *hypersurface* $y = -\frac{1}{\nu}$. This clearly makes the metric singular and constant time slicing gives $S^1 \otimes S^2$ topology for its cross-section. Lets compute its normal vector field. Consider, for some $\tilde{f} \epsilon \mathcal{F}$;

$$\begin{aligned} l &= \tilde{f} g^{\mu\nu} \frac{\partial y}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \\ &= \tilde{f} g^{yy} \frac{\partial}{\partial y} \\ So,\ l^\mu l_\mu &= l^y l_y = \tilde{f}^2 g^{yy} \\ Hence,\ |l^\mu l_\mu|_{y=-\frac{1}{\nu}} &= 0\ (as\ g^{yy}\ vanishes\ here) \end{aligned}$$

So, $y = -\frac{1}{\nu}$ is indeed a null *hypersurface*. Hence, from the above discussions we conclude that, $y = -\frac{1}{\nu}$ is indeed the event horizon of the black ring.

Now, we would compute the *killing vector field*. Observing the metric we see that $g^{\mu\nu}$ is independent of (t, ϕ, ψ), so; the three linearly independent *killing vector fields* we get from this observation are :-

$$\xi_1 = \frac{\partial}{\partial t} \tag{6.129}$$

$$\xi_2 = \frac{\partial}{\partial \phi} \tag{6.130}$$

$$\xi_3 = \frac{\partial}{\partial \psi} \tag{6.131}$$

So, we choose the killing vector field to be :-

$$\xi = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial \phi} + \beta \frac{\partial}{\partial \psi}$$

So, $\xi^t = 1, \xi^\phi = \alpha, \xi^\psi = \beta$. Now,

$$\begin{aligned} \xi^\mu \xi_\mu &= g_{\mu\nu} \xi^\mu \xi_\nu = g_{tt} + 2g_{t\phi}\alpha + 2g_{t\psi}\beta + 2g_{\phi\psi}\alpha\beta + g_{\phi\phi}\alpha^2 + g_{\psi\psi}\beta^2 \\ &= -\frac{F(y)}{F(x)} + \frac{2CR(1+y)}{F(x)}\beta + \frac{R^2 G(x)}{(x-y)^2}\alpha^2 - \left(\frac{C^2 R^2 (1+y)^2}{F(x)F(y)} + \frac{R^2 F(x) G(y)}{(x-y)^2 F(y)} \right) \beta^2 \end{aligned}$$

Now as before we find β by demanding; $|\xi^\mu \xi_\mu|_{y=-\frac{1}{\nu}} = 0$. This gives at $y = -\frac{1}{\nu}$:-

$$-F(y) + 2CR(1+y)\beta + \frac{R^2 G(x) F(x)}{(x-y)^2}\alpha^2 - \frac{C^2 R^2 (1+y)^2}{F(y)}\beta^2 = 0$$

Since, α and β should be constant they can't be functions of x. So, $\alpha = 0$ as only its coefficients contains functions of x. Hence,

$$\xi = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial\psi}$$

So, we obtain :-

$$\begin{aligned}
&- F\left(-\frac{1}{\nu}\right) + 2CR\left(1-\frac{1}{\nu}\right)\beta - C^2R^2\left(1-\frac{1}{\nu}\right)^2\frac{\beta^2}{F\left(-\frac{1}{\nu}\right)} = 0\\
\Rightarrow\ & -1 + \frac{\lambda}{\nu} + 2CR\left(1-\frac{1}{\nu}\right)\beta - C^2R^2\left(1-\frac{1}{\nu}\right)^2\frac{\beta^2}{\left(1-\frac{\lambda}{\nu}\right)} = 0\\
\Rightarrow\ & \lambda - \nu + 2CR(\nu-1)\beta - C^2R^2\frac{(\nu-1)^2\beta^2}{\nu-\lambda} = 0\\
\Rightarrow\ & C^2R^2(\nu-1)^2\beta^2 + 2CR(\nu-1)(\lambda-\nu)\beta + (\lambda-\nu)^2 = 0\\
\Rightarrow\ & \beta = \frac{\lambda-\nu}{CR(1-\nu)} = \frac{1}{R(1-\nu)}\left(\frac{(\lambda-\nu)(1-\lambda)}{\lambda(1+\lambda)}\right)^{\frac{1}{2}}
\end{aligned}$$

So, this makes; $y = -\frac{1}{\nu}$ a *killing horizon* for the *killing vector field* ξ with ξ given as :-

$$\xi = \frac{\partial}{\partial t} + \frac{1}{R(1-\nu)}\left(\frac{(\lambda-\nu)(1-\lambda)}{\lambda(1+\lambda)}\right)^{\frac{1}{2}}\frac{\partial}{\partial\psi} \tag{6.132}$$

Now, using $\nabla_\alpha(-\xi^\beta\xi_\beta) = 2\kappa\xi_\alpha$; we get :-

$$\begin{aligned}
(-\xi^\mu\xi_\mu)_{;\alpha} = &\frac{\lambda}{1+\lambda x}\partial_\alpha y - \frac{\lambda(1+\lambda y)}{(1+\lambda x)^2}\partial_\alpha x - \frac{2}{1+\lambda x}\left(\frac{\lambda-\nu}{1-\nu}\right)\partial_\alpha y\\
&+ \frac{2\lambda(1+y)}{(1+\lambda x)^2}\left(\frac{\lambda-\nu}{1-\nu}\right)\partial_\alpha x + \frac{2(1+y)}{(1+\lambda x)(1+\lambda y)}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\partial_\alpha y - \frac{\lambda(1+y)^2}{(1+\lambda y)^2(1+\lambda x)}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\partial_\alpha y\\
&- \frac{\lambda(1+y)^2}{(1+\lambda y)(1+\lambda x)^2}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\partial_\alpha x\\
&+ \frac{1+\lambda x}{(x-y)^2(1+\lambda y)}\frac{(\lambda-\nu)(1-\lambda)}{\lambda(1+\lambda)(1-\nu)}[(1-y^2)\nu\ +\ terms\ with\ (1+\nu y)]\partial_\alpha y\\
&+\ terms\ with\ (1+\nu y)\partial_\alpha y\\
Now, |(-\xi^\mu\xi_\mu)_{;\alpha}|_{y=-\frac{1}{\nu}} = &\frac{\lambda}{1+\lambda x}\partial_\alpha y - \frac{\lambda}{(1+\lambda x)^2}\left(1-\frac{\lambda}{\nu}\right)\partial_\alpha x - \frac{2}{1+\lambda x}\left(\frac{\lambda-\nu}{1-\nu}\right)\partial_\alpha y\\
&+ \frac{2\lambda\left(1-\frac{1}{\nu}\right)}{(1+\lambda x)^2}\left(\frac{\lambda-\nu}{1-\nu}\right)\partial_\alpha x + \frac{2\left(1-\frac{1}{\nu}\right)}{(1+\lambda x)\left(1-\frac{\lambda}{\nu}\right)}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\partial_\alpha y - \frac{\lambda\left(1-\frac{1}{\nu}\right)^2}{\left(1-\frac{\lambda}{\nu}\right)^2(1+\lambda x)}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\partial_\alpha y\\
&- \frac{\lambda\left(1-\frac{1}{\nu}\right)^2}{\left(1-\frac{\lambda}{\nu}\right)^2(1+\lambda x)^2}\left(\frac{\lambda-\nu}{1-\nu}\right)\partial_\alpha x + \frac{1+\lambda x}{\left(x+\frac{1}{\nu}\right)^2\left(1-\frac{\lambda}{\nu}\right)}\frac{(\lambda-\nu)(1-\lambda)}{\lambda(1+\lambda)(1-\nu)}\nu\left(1-\frac{1}{\nu^2}\right)\partial_\alpha y\\
= &\frac{1+\lambda x}{(1+\nu x)^2F(-\frac{1}{\nu})}\frac{1}{C}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\nu\left(1-\frac{1}{\nu^2}\right)\partial_\alpha y
\end{aligned}$$

Now, $\xi_\alpha = c_1\partial_\alpha y$, where; $(c_1\epsilon\Re)$ and $c_1^2 = |g_{yy}\xi^\mu\xi_\mu|_{y=-\frac{1}{\nu}}$. Now, since we know that at $y = -\frac{1}{\nu}$ both $G(y)$ and $\xi^\mu\xi_\mu$ vanish so taking limit $y \to -\frac{1}{\nu}$ and applying L-Höpital's rule we finally observe that the term containing $G(y)$ in $\xi^\mu\xi_\mu$ survives and so;

$$\begin{aligned}
c_1^2 &= \frac{R^2F(x)^2}{(x-y)^4F(-\frac{1}{\nu})}\frac{1}{C^2}\left(\frac{\lambda-\nu}{1-\nu}\right)^2\\
c_1 &= \frac{RF(x)}{(x-y)^2F(-\frac{1}{\nu})^{\frac{1}{2}}}\frac{1}{C}\left(\frac{\lambda-\nu}{1-\nu}\right)
\end{aligned}$$

Now, $|\nabla_\alpha(-\xi^\mu\xi_\mu)|_{y=-\frac{1}{\nu}} = |2\kappa\xi_\alpha| = |2\kappa c_1\partial_\alpha y|$ gives :-

$$2\kappa R = \frac{\nu^{\frac{1}{2}}}{(\nu-\lambda)^{\frac{1}{2}}}\frac{(1+\nu)(\lambda-\nu)}{\nu}\left(\frac{1-\lambda}{\lambda(\lambda-\nu)(1+\lambda)}\right)^{\frac{1}{2}}$$

Hence, we obtain :-

$$\kappa = \frac{1+\nu}{2R}\left(\frac{1-\lambda}{\nu\lambda(1+\lambda)}\right)^{\frac{1}{2}} \tag{6.133}$$

$$T_H = \frac{1+\nu}{4\pi R}\left(\frac{1-\lambda}{\nu\lambda(1+\lambda)}\right)^{\frac{1}{2}} \tag{6.134}$$

6.7.2 Charged Black Ring

Following similar coordinate construction, we present the metric for the charged rotating black ring :-

$$ds^2 = -\frac{F(y)}{F(x)}\left(\frac{H(x)}{H(y)}\right)^{\frac{N}{3}}\left(dt + CR\frac{1+y}{F(y)}d\psi\right)^2 \tag{6.135}$$

$$+ \frac{R^2}{(x-y)^2}F(x)(H(x)H(y)^2)^{\frac{N}{3}}\left[-\frac{G(y)}{F(y)H(y)^N}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)H(x)^N}d\phi^2\right] \tag{6.136}$$

where $F(z) = 1+\lambda z; G(z) = (1-z^2)(1+\nu z); H(z) = 1-\mu z; C = \left[\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}\right]^{\frac{1}{2}}$, with; $(0 < \nu \leq \lambda < 1)$ and $(0 \leq \mu < 1)$. Furthermore, R has dimensions of length, and for thin large rings it corresponds roughly to the radius of the ring circle and μ is a parameter related to local charge $\mathcal{Q}$ of the ring.

Here again to remove the conical singularity we have the relation :-

$$\frac{1-\lambda}{1+\lambda}\left(\frac{1+\mu}{1-\mu}\right)^N = \left(\frac{1-\nu}{1+\nu}\right)^2 \tag{6.137}$$

Now, putting $\mu = \nu = \lambda = 0$ we get back the flat metric in the form of eq^n 6.121 . Also, the ring source's location is at $y = -\infty$. Here also we would choose (motivated by previous case) the family of *hypersurfaces* to be $y = constant$ which would contain horizons. $y = -\frac{1}{\nu}$ is the *hypersurface* describing the event horizon. Now let us compute the normal vector fields. Consider, for some $\tilde{f}\epsilon\mathcal{F}$;

$$\begin{aligned} l &= \tilde{f}g^{\mu\nu}\frac{\partial y}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \\ &= \tilde{f}g^{yy}\frac{\partial}{\partial y} \\ So,\ l^\mu l_\mu &= l^y l_y = \tilde{f}^2 g^{yy} \\ Hence,\ |l^\mu l_\mu|_{y=-\frac{1}{\nu}} &= 0\ (as\ g^{yy}\ vanishes\ here) \end{aligned}$$

So, $y = -\frac{1}{\nu}$ is indeed a null *hypersurface*. Hence, from the above discussions we conclude that, $y = -\frac{1}{\nu}$ is indeed the event horizon of the black ring.

Now, we would compute the *killing vector field*. Observing the metric we see that $g^{\mu\nu}$ is independent of (t, ϕ, ψ), so; the three linearly independent *killing vector fields* we get from this observation are :-

$$\xi_1 = \frac{\partial}{\partial t} \tag{6.138}$$

$$\xi_2 = \frac{\partial}{\partial \phi} \tag{6.139}$$

$$\xi_3 = \frac{\partial}{\partial \psi} \tag{6.140}$$

So, we choose the killing vector field to be :-

$$\xi = \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial \phi} + \beta\frac{\partial}{\partial \psi}$$

So, $\xi^t = 1, \xi^\phi = \alpha, \xi^\psi = \beta$. Now,

$$\begin{aligned} \xi^\mu\xi_\mu &= g_{tt} + 2g_{t\psi}\beta + g_{\phi\phi}\alpha^2 + g_{\psi\psi}\beta^2 \\ &= -\frac{F(y)}{F(x)}\left(\frac{H(x)}{H(y)}\right)^{\frac{N}{3}} - \frac{2CR(1+y)}{F(x)}\left(\frac{H(x)}{H(y)}\right)^{\frac{N}{3}}\beta + \frac{R^2(H(x)H(y)^2)^{\frac{N}{3}}}{(x-y)^2H(x)^N}G(x)\alpha^2 - \frac{C^2R^2(1+y)^2}{F(y)F(x)}\left(\frac{H(x)}{H(y)}\right)^{\frac{N}{3}}\beta^2 \\ &- \frac{R^2F(x)(H(x)H(y)^2)^{\frac{N}{3}}}{(x-y)^2}\frac{G(y)}{F(y)H(y)^N}\beta^2 \end{aligned}$$

Now as before we find β by demanding; $|\xi^\mu\xi_\mu|_{y=-\frac{1}{\nu}} = 0$. This gives at $y = -\frac{1}{\nu}$:-

$$-F(y)\left(\frac{1}{H(y)}\right)^{\frac{N}{3}} - 2CR(1+y)\left(\frac{1}{H(y)}\right)^{\frac{N}{3}}\beta + \frac{R^2F(x)(H(y)^2)^{\frac{N}{3}}}{(x-y)^2H(x)^N}G(x)\alpha^2 - \frac{C^2R^2(1+y)^2}{F(y)}\left(\frac{1}{H(y)}\right)^{\frac{N}{3}}\beta^2 = 0$$

Since, α and β should be constant they can't be functions of x. So, $\alpha = 0$ as only its coefficients contains functions of x. Hence,

$$\xi = \frac{\partial}{\partial t} + \beta\frac{\partial}{\partial\psi}$$

So, we obtain :-

$$\begin{aligned}
&-F(y) - 2CR(1+y)\beta - \frac{C^2R^2(1+y)^2}{F(y)}\beta^2 = 0 \\
&\Rightarrow C^2R^2(1+y)^2\beta^2 + 2CRF(y)(1+y)\beta + F(y)^2 = 0 \\
&\Rightarrow \beta = \frac{\lambda-\nu}{CR(\nu-1)}
\end{aligned}$$

So, this makes; $y = -\frac{1}{\nu}$ a *killing horizon* for the *killing vector field* ξ with ξ given as :-

$$\xi = \frac{\partial}{\partial t} + \frac{\lambda-\nu}{CR(\nu-1)}\frac{\partial}{\partial\psi} \tag{6.141}$$

Now, using $\nabla_\alpha(-\xi^\beta\xi_\beta) = 2\kappa\xi_\alpha$; we get :-

$$\begin{aligned}
(-\xi^\mu\xi_\mu)_{;\alpha} = &\frac{\lambda}{1+\lambda x}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\partial_\alpha y - \frac{\lambda(1+\lambda y)}{(1+\lambda x)^2}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\partial_\alpha x - \frac{\mu(1+\lambda y)}{1+\lambda x}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}-1}}{(1-\mu y)^{\frac{N}{3}}}\partial_\alpha x \\
&+ \frac{\mu(1+\lambda y)}{1+\lambda x}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}}}{(1-\mu y)^{\frac{N}{3}+1}}\partial_\alpha y + \frac{2}{1+\lambda x}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\left(\frac{\lambda-\nu}{\nu-1}\right)\partial_\alpha y - \frac{2\lambda(1+y)}{(1+\lambda x)^2}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\left(\frac{\lambda-\nu}{\nu-1}\right)\partial_\alpha x \\
&- \frac{2\mu(1+y)}{1+\lambda x}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}-1}}{(1-\mu y)^{\frac{N}{3}}}\left(\frac{\lambda-\nu}{\nu-1}\right)\partial_\alpha x + \frac{2\mu(1+y)}{1+\lambda x}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}}}{(1-\mu y)^{\frac{N}{3}+1}}\left(\frac{\lambda-\nu}{\nu-1}\right)\partial_\alpha y \\
&+ \frac{2(1+y)}{(1+\lambda x)(1+\lambda y)}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\left(\frac{\lambda-\nu}{\nu-1}\right)^2\partial_\alpha y - \frac{\lambda(1+y)^2}{(1+\lambda x)^2(1+\lambda y)}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\left(\frac{\lambda-\nu}{\nu-1}\right)^2\partial_\alpha x \\
&- \frac{\lambda(1+y)^2}{(1+\lambda x)(1+\lambda y)^2}\left(\frac{1-\mu x}{1-\mu y}\right)^{\frac{N}{3}}\left(\frac{\lambda-\nu}{\nu-1}\right)^2\partial_\alpha y - \frac{\mu(1+y)^2}{(1+\lambda x)(1+\lambda y)}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}-1}}{(1-\mu y)^{\frac{N}{3}}}\left(\frac{\lambda-\nu}{\nu-1}\right)^2\partial_\alpha x \\
&+ \frac{\mu(1+y)^2}{(1+\lambda x)(1+\lambda y)}\frac{N}{3}\frac{(1-\mu x)^{\frac{N}{3}}}{(1-\mu y)^{\frac{N}{3}+1}}\left(\frac{\lambda-\nu}{\nu-1}\right)^2\partial_\alpha y + \frac{1+\lambda x}{(x-y)^2(1+\lambda y)}\frac{(1-\mu x)^{\frac{N}{3}}}{(1-\mu y)^{\frac{N}{3}+1}}\frac{1}{C^2} \\
&\times [(1-y^2)\nu\ +\ terms\ with\ (1+\nu y)\partial_\alpha y]\ +\ terms\ with\ (1+\nu y)\partial_\alpha y
\end{aligned}$$

Now, at $y = -\frac{1}{\nu}$; after a messy calculation we obtain :-

$$|(-\xi^\mu\xi_\mu)_{;\alpha}|_{y=-\frac{1}{\nu}} = \frac{1+\lambda x}{(1+\nu x)^2}\frac{\nu^2}{C^2}\left(\frac{1-\mu x}{1+\frac{\mu}{\nu}}\right)^{\frac{N}{3}}\left(\frac{\nu-\lambda}{\nu-1}\right)(\nu+1)\partial_\alpha y$$

Now, $\xi_\alpha = c_1\partial_\alpha y$, where; ($c_1\epsilon\Re$) and $c_1^2 = |g_{yy}\xi^\mu\xi_\mu|_{y=-\frac{1}{\nu}}$. Now, since we know that at $y = -\frac{1}{\nu}$ both $G(y)$ and $\xi^\mu\xi_\mu$ vanish so taking limit $y \to -\frac{1}{\nu}$ and applying L-Höpital's rule we finally observe that the term containing $G(y)$ in $\xi^\mu\xi_\mu$ survives and so at $y = -\frac{1}{\nu}$;

$$\begin{aligned}
c_1^2 &= \frac{R^2F(x)^2}{(x-y)^4}\frac{H(x)^{\frac{2N}{3}}H(y)^{\frac{N}{3}}}{1+\lambda y}\frac{1}{C^2}\left(\frac{\lambda-\nu}{\nu-1}\right)^2 \\
c_1 &= \frac{RF(x)}{(x-y)^2}\frac{H(x)^{\frac{N}{3}}H(y)^{\frac{N}{6}}}{(1+\lambda y)^{\frac{1}{2}}}\frac{1}{C}\left(\frac{\lambda-\nu}{\nu-1}\right)
\end{aligned}$$

Now, $|\nabla_\alpha(-\xi^\mu\xi_\mu)|_{y=-\frac{1}{\nu}} = |2\kappa\xi_\alpha| = |2\kappa c_1\partial_\alpha y|$ gives :-

$$2\kappa R = \frac{1}{R}\frac{1}{(1+\frac{\mu}{\nu})^{\frac{1}{2}}}\frac{1+\nu}{\nu^{\frac{1}{2}}}\frac{(1-\lambda)^{\frac{1}{2}}}{(1+\lambda)^{\frac{1}{2}}\lambda^{\frac{1}{2}}}$$

Hence, we obtain :-

$$\kappa = \frac{1+\nu}{2R} \frac{\nu^{\left(\frac{N-1}{2}\right)}}{(\mu+\nu)^{\frac{N}{2}}} \left(\frac{1-\lambda}{\lambda(1+\lambda)}\right)^{\frac{1}{2}} \tag{6.142}$$

$$T_H = \frac{1+\nu}{4\pi R} \frac{\nu^{\left(\frac{N-1}{2}\right)}}{(\mu+\nu)^{\frac{N}{2}}} \left(\frac{1-\lambda}{\lambda(1+\lambda)}\right)^{\frac{1}{2}} \tag{6.143}$$

Taking $\mu = 0$ in $eq^n s$ 6.142 and 6.143 gives back $eq^n s$ 6.133 and 6.134 .

Future Prospects

After this project work there are various other related works left to be done in this summer which could not be done due to the time constraint of the summer project. These works include :-

1. Lagrangian and Hamiltonian Formulation of General Relativity
2. Justifying the name surface gravity using Rindler spacetime observer
3. Remaining Laws of Black Hole Mechanics including Smarr's Formula
4. The Quantum Field Theoretic proof of $T_H = \frac{\kappa}{2\pi}$
5. Hawking-Bekenstein radiation
6. Singularity Theorems
7. Wald Entropy

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