Problem Set 1 - General Relativity - Date: 26/01/2018

1. From Newton to Maxwell

Consider Galilean transformations between two frames $\mathcal{O}$ and $\mathcal{O}'$ (where, $\mathcal{O}'$ frame is moving with uniform velocity v_0 w.r.t. $\mathcal{O}$ along x-axis). For simplicity, we will just focus on one spatial dimension.

$$x' = x - v_0 t \quad , \quad t' = t \tag{1}$$

Now let us look at Maxwell's equations in the $\mathcal{O}$ frame :-

$$\overline{\nabla} \cdot \overline{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{2}$$

$$\overline{\nabla} \cdot \overline{\mathbf{B}} = 0 \tag{3}$$

$$\overline{\nabla} \times \overline{\mathbf{E}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t} \tag{4}$$

$$\overline{\nabla} \times \overline{\mathbf{B}} = \mu_0 \overline{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \overline{\mathbf{E}}}{\partial t} \tag{5}$$

In vacuum (i.e., source free) the above equations become :-

$$\overline{\nabla} \cdot \overline{\mathbf{E}} = 0 \tag{6}$$

$$\overline{\nabla} \cdot \overline{\mathbf{B}} = 0 \tag{7}$$

$$\overline{\nabla} \times \overline{\mathbf{E}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t} \tag{8}$$

$$\overline{\nabla} \times \overline{\mathbf{B}} = \mu_0 \epsilon_0 \frac{\partial \overline{\mathbf{E}}}{\partial t} \tag{9}$$

a) Taking the curl of $eq^n(8)$ and $eq^n(9)$ arrive at the following wave equations for the $\mathcal{O}$ frame[1] :-

$$\nabla^2 \overline{\mathbf{E}} = \mu_0 \epsilon_0 \frac{\partial^2 \overline{\mathbf{E}}}{\partial t^2} \tag{10}$$

$$\nabla^2 \overline{\mathbf{B}} = \mu_0 \epsilon_0 \frac{\partial^2 \overline{\mathbf{B}}}{\partial t^2} \tag{11}$$

b) Now since light is an already established electromagnetic wave, take a *leap of faith* and identify the speed of light, c, in the $\mathcal{O}$ frame, from $eq^n(10)$ or, $eq^n(11)$. What does this identification suggests about the nature of c in the context of the $\mathcal{O}'$ frame?

Now let us look at a general wave equation of the form $eq^n(10)$ or, $eq^n(11)$ in one spatial dimension, with $\Psi(x,t) = \overline{\mathbf{E}}$ or, $\overline{\mathbf{B}}$. We have :-

$$-\frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} + \frac{\partial^2 \Psi}{\partial x^2} = 0 \tag{12}$$

c) Check that $\Psi(t,x) = \Psi(x - ct)$ is a solution $eq^n(12)$.

d) Derive the form of $eq^n(12)$ in the $\mathcal{O}'$ frame using the Galilean transformations. Does the form of wave equation in $\mathcal{O}$ frame and $\mathcal{O}'$ frame match?

e) Check that $\Psi(t', x') = \Psi(x' - (c - v_0)t')$ is a solution of the wave equation in $\mathcal{O}'$ frame[2]

2. The Minkowski metric

Let us work in natural units from now on where $c = 1$. Consider the Minkowski metric :-

$$ds_M^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \tag{14}$$

[1] Hint: Use the identity, $\overline{\nabla} \times (\overline{\nabla} \times \overline{\mathbf{A}}) = \overline{\nabla}(\overline{\nabla} \cdot \overline{\mathbf{A}}) - \nabla^2 \overline{\mathbf{A}}$

[2] The wave equation in the $\mathcal{O}'$ frame is :-

$$-\frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t'^2} + \frac{2v_0}{c^2}\frac{\partial^2 \Psi}{\partial x' \partial t'} + \left(1 - \frac{v_0}{c^2}\right)\frac{\partial^2 \Psi}{\partial x'^2} = 0 \tag{13}$$

Check whether your answer for **d)** matches with $eq^n(13)$.

It was already shown in the class that ds_M^2 is invariant under *Lorentz boost*[3]

a) Show that under translation, i.e., $x' = x - d$, ds_M^2 is invariant[4].

b) Show that under spatial rotation in the xy-plane, ds_M^2 is invariant[5].

c) Show that under parity transformation, $\mathbf{P} = diag(1, -1, -1, -1)$ and under time reversal transformation, $\mathbf{T} = diag(-1, 1, 1, 1)$, ds_M^2 remains invariant.

3. Spacelike vectors

We know that in the 4-vector notation, we have the 4-velocity of a particle (in the $\mathcal{O}$ frame) as[6] :-

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(\gamma, \gamma\frac{dx}{dt}, \gamma\frac{dy}{dt}, \gamma\frac{dz}{dt}\right) \tag{16}$$

a) Compute the 4-acceleration, $a^\mu = \frac{du^\mu}{d\tau}$. Compute its norm and establish $a^\mu a_\mu < 0$ thereby, proving that a^μ is a spacelike vector[7].

4. Metric inverse

Consider any arbitrary metric $g_{\mu\nu}$. We know it is a $rank-(0,2)$ tensor and it can be viewed as a 4×4 matrix. Let us define its inverse matrix as :-

$$(g_{\mu\nu})^{-1} := g^{\mu\nu} \tag{17}$$

a) Starting from the relation, $g^{\mu\alpha} g_{\mu\beta} = \delta^\alpha_\beta$, which is nothing but the definition of an inverse matrix, show that $g^{\mu\nu}$ is a $rank-(2,0)$ tensor.

5. Poincare transformations as *smooth* Linear transformations

Recall that we could write Lorentz transformations as :-

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{18}$$

From **2.a**), we see that translations also preserve the invariance of the Minkowski metric. So, the most general transformations under the domain of Special Relativity, called the Poincare transformations are given as :-

$$x'^\mu = \Lambda^\mu_\nu x^\nu + d^\mu \tag{19}$$

where d^μ is a constant 4-vector which describes translations.

a) Derive the transformation rule for $\eta_{\mu\nu}$ by demanding that ds_M^2 remains invariant under Poincare transformations $eq^n(18)$[8].

Let us see what is the nature of the Poincare transformations. For generality, let us assume $\eta_{\mu\nu}$ are the most general coordinate transformations leaving ds_M^2 invariant and hence each component of $\eta_{\mu\nu}$ is a constant function of the coordinate, x^μ. Thus[9],

$$\eta_{\alpha\beta} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}\eta'_{\mu\nu} \tag{20}$$

[3] *Lorentz boost* is the following transformation :-

$$t' = \gamma\left(t - \frac{v_0 x}{c^2}\right) \quad , \quad x' = \gamma(x - v_0 t) \quad , \quad y' = y \quad , \quad z' = z \tag{15}$$

where, $\gamma = \frac{1}{\sqrt{1 - v_0^2/c^2}}$.

[4] translation is only along x-axis

[5] Hint: Use the rotation matrix for the computation

[6] where, $d\tau := \frac{ds_M}{c} = ds_M$ (in natural units).

[7] Hint: Use the coordinate time t and proper time τ relation, $dt = \gamma d\tau$, and compute a^μ component by component. This relation is nothing other than the usual time dilation relation. Also, do not forget to differentiate γ while differentiating u^μ w.r.t. τ. Convince yourself that γ is not a constant

[8] Hint: For arbitrary metric, $g_{\mu\nu}$ this exercise was done in the class and that is how we had obtained the transformation rule for $g_{\mu\nu}$.

[9] Recall from class the transformation rule for $g_{\mu\nu}$. Here also $\eta_{\mu\nu}$ has exactly the same transformation but in this case entries of $\eta_{\mu\nu}$ are constants.

b) Now differentiate $eq^n(19)$ w.r.t. x^λ to obtain :-

$$0 = \frac{\partial \eta_{\alpha\beta}}{\partial x^\lambda} = \frac{\partial^2 x'^\mu}{\partial x^\alpha x^\lambda}\frac{\partial x'^\nu}{\partial x^\beta}\eta'_{\mu\nu} + \frac{\partial^2 x'^\nu}{\partial x^\beta x^\lambda}\frac{\partial x'^\mu}{\partial x^\alpha}\eta'_{\mu\nu} \tag{21}$$

c) Now, interchange λ with β in $eq^n(20)$ to get :-

$$0 = \frac{\partial \eta_{\alpha\lambda}}{\partial x^\beta} = \frac{\partial^2 x'^\mu}{\partial x^\alpha x^\beta}\frac{\partial x'^\nu}{\partial x^\lambda}\eta'_{\mu\nu} + \frac{\partial^2 x'^\nu}{\partial x^\beta x^\lambda}\frac{\partial x'^\mu}{\partial x^\alpha}\eta'_{\mu\nu} \tag{22}$$

d) Now, interchange λ with α in $eq^n(20)$ to get :-

$$0 = \frac{\partial \eta_{\lambda\beta}}{\partial x^\alpha} = \frac{\partial^2 x'^\mu}{\partial x^\alpha x^\lambda}\frac{\partial x'^\nu}{\partial x^\beta}\eta'_{\mu\nu} + \frac{\partial^2 x'^\nu}{\partial x^\beta x^\alpha}\frac{\partial x'^\mu}{\partial x^\lambda}\eta'_{\mu\nu} \tag{23}$$

e) Now perform $eq^n(20) + eq^n(21) - eq^n(22)$ to obtain :-

$$2\eta'_{\mu\nu}\frac{\partial^2 x'^\nu}{\partial x^\lambda \partial x^\beta}\frac{\partial x'^\mu}{\partial x^\alpha} = 0 \tag{24}$$

f) Multiply $eq^n(23)$ by $\eta'^{\sigma\mu}$ to get :-

$$\frac{\partial^2 x'^\sigma}{\partial x^\lambda \partial x^\beta}\frac{\partial x'^\mu}{\partial x^\alpha} = 0 \tag{25}$$

g) Since, $\frac{\partial x'^\mu}{\partial x^\alpha} \neq 0$ we get :-

$$\frac{\partial^2 x'^\sigma}{\partial x^\lambda \partial x^\beta} = 0 \tag{26}$$

$Eq^n(25)$ shows that Poincare transformations are the most general smooth linear transformations.

6. Null vectors

a) Show that a vector orthogonal to an arbitrary null vector is a null vector or a spacelike vector[10].

7. Geodesics on a plane

Consider the two-dimensional Euclidean metric written in polar coordinates (r, θ)[11] :-

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{27}$$

a) Set up the geodesic equation for the above metric.

8. Parallel transport on plane and on S^1

9. Motivation for the postulate of Riemannian geometry

10. 4-vector transformation rule from old school knowledge

[10] An arbitrary null vector A^μ satisfies $A^\mu A_\mu = 0$. So, do not just consider the trivial null vector whose all components are zero for the problem.

[11] There is nothing to find geodesics on a plane in Cartesian coordinates. They are clearly straight lines since the metric components are all constants and hence from the geodesic equation this is utterly obvious.