2D Dilaton Gravity and Moving Mirrors.

Summer Project Report

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Abstract

In this project we have studied 2D dilaton gravity model coupled to massless scalar fields in various contexts. First, we studied the eternal 2D dilaton black hole solutions. Then, we considered the case of 2D dilaton gravity model with a dynamical boundary term. We noted the similarity in the *Carter-Penrose* diagram of the latter with that of the collapsing spherically symmetric (3+1) dimensional matter. Furthermore, we considered the phenomenon of Hawking radiation in the 2D dilaton gravity model without a boundary case. Lastly, we gave a brief introduction to the moving mirror models which can serve as a toy model for the Hawking radiation process.

Notations for *Carter-Penrose* Diagram

We give here some important notations which we will use in various *Carter-Penrose* diagrams. Here we give notations with reference to (1+1) dimensional Minkowski spacetime. Here,

$$\tilde{v} = t + x$$
$$\tilde{u} = t - x$$
$$\tilde{v} = \tan \hat{v}$$
$$\tilde{u} = \tan \hat{u}$$

Then,

$$Right\ spacelike\ infinity, \quad i^0_R \rightarrow \begin{cases} \hat{u} = -\frac{\pi}{2} & \tilde{u} \rightarrow -\infty \quad x \rightarrow \infty \\ \hat{v} = \frac{\pi}{2} & \tilde{v} \rightarrow \infty \quad\ \ t \ \ finite \end{cases}$$

$$Left\ spacelike\ infinity, \quad i^0_L \rightarrow \begin{cases} \hat{u} = \frac{\pi}{2} & \tilde{u} \rightarrow \infty \quad\ \ x \rightarrow -\infty \\ \hat{v} = -\frac{\pi}{2} & \tilde{v} \rightarrow -\infty \quad t \ \ finite \end{cases}$$

$$Past\ timelike\ infinity, \quad i^- \rightarrow \begin{cases} \hat{u} = -\frac{\pi}{2} & \tilde{u} \rightarrow -\infty \quad x \ \ finite \\ \hat{v} = -\frac{\pi}{2} & \tilde{v} \rightarrow -\infty \quad t \rightarrow -\infty \end{cases}$$

$$Future\ timelike\ infinity, \quad i^+ \rightarrow \begin{cases} \hat{u} = \frac{\pi}{2} & \tilde{u} \rightarrow \infty \quad x \ \ finite \\ \hat{v} = \frac{\pi}{2} & \tilde{v} \rightarrow \infty \quad t \rightarrow \infty \end{cases}$$

$$Right\ past\ nulllike\ infinity, \quad \mathcal{J}^-_{\mathcal{R}} \rightarrow \begin{cases} \hat{u} = -\frac{\pi}{2} & \tilde{u} \rightarrow -\infty \quad x \rightarrow \infty \\ |\hat{v}| \neq \frac{\pi}{2} & \tilde{v} \ \ finite \quad t \rightarrow -\infty \end{cases}$$

$$Right\ future\ nulllike\ infinity, \quad \mathcal{J}^+_{\mathcal{R}} \rightarrow \begin{cases} |\hat{u}| \neq \frac{\pi}{2} & \tilde{u} \ \ finite \quad x \rightarrow \infty \\ \hat{v} = \frac{\pi}{2} & \tilde{v} \rightarrow \infty \quad\ \ t \rightarrow \infty \end{cases}$$

$$Left\ past\ nulllike\ infinity, \quad \mathcal{J}^-_{\mathcal{L}} \rightarrow \begin{cases} |\hat{u}| \neq \frac{\pi}{2} & \tilde{u} \ \ finite \quad x \rightarrow -\infty \\ \hat{v} = -\frac{\pi}{2} & \tilde{v} \rightarrow -\infty \quad t \rightarrow -\infty \end{cases}$$

$$Left\ future\ nulllike\ infinity, \quad \mathcal{J}^+_{\mathcal{L}} \rightarrow \begin{cases} \hat{u} = \frac{\pi}{2} & \tilde{u} \rightarrow \infty \quad\ \ x \rightarrow -\infty \\ |\hat{v}| \neq \frac{\pi}{2} & \tilde{v} \ \ finite \quad t \rightarrow \infty \end{cases}$$

Contents

Chapter 1

2D Dilaton Gravity Model

The 2D Dilaton gravity models are models in (1+1) dimensional spacetime which give insights into understanding the Hawking radiation phenomenon. It is a simple model which is analytically solvable and also for which the Hawking radiation calculations are exactly obtainable in closed form. Due to these and various other reasons associated in understanding the Hawking radiation phenomenon, this model has been studied for quiet some time starting with Callan, Giddings, Harvey and Strominger (CGHS), [1]. So, this is our motivation to study this model since we will be trying to understand the Hawking radiation process better.

First, we will motivate how to arrive at the action for the 2D Dilaton gravity model.

1.1 The CGHS Lagrangian

1.1.1 Setting up the Lagrangian

Consider a generally spherically symmetric metric in (3+1) dimensions of the form :-

$$ds^2_{(4)} = g_{\mu\nu}dx^\mu dx^\nu + \frac{e^{-2\phi}}{\lambda^2}d\Omega_2^2 \tag{1.1}$$

where, $\mu, \nu = 0, 1$; $\phi, g_{\mu\nu}$ are functions of $(x^0, x^1) = (t, r)$ and λ is a constant.

Now, plugging $eq^n(1.1)$ in the Einstein-Hilbert action, $S_{E-H} = \int d^4x\sqrt{-g}\frac{R^{(4)}}{16\pi G_N}$ and integrating out θ, φ we get in units with $G_N = \frac{\pi}{2\lambda^2}$:-

$$S_{reduced} = \frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R + 2(\nabla\phi)^2 + 2\lambda^2 e^{2\phi}] \tag{1.2}$$

Let us now introduce the CGHS action, [1] which is the classical 2D Dilaton gravity action :-

$$S_D = \frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R + 4(\nabla\phi)^2 + 4\lambda^2] \tag{1.3}$$

where, ϕ is called the dilaton field.

Note that, $eq^n(1.3)$ differs from $eq^n(1.2)$ in the numerical coefficient of the dilaton kinetic energy term and the ϕ-dependence of the potential. The reason behind studying $eq^n(1.3)$ rather than $eq^n(1.2)$ is that the theory described by $eq^n(1.3)$ is dramatically simpler to study; the classical solutions can be presented in explicit closed form. The action $eq^n(1.3)$ arises in a low-energy effective description of certain dilatonic black holes in string theory, [8].

1.1.2 Equations of motion and Vacuum of the model

Variations of the action S_D gives the following equations of motion :-

$$2e^{-2\phi}[\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}((\nabla\phi)^2 - \nabla^2\phi - \lambda^2)] = 0 \quad (variation\ w.r.t.\ g_{\mu\nu}) \tag{1.4}$$

$$2e^{-2\phi}[R + 4\lambda^2 + 4\nabla^2\phi - 4(\nabla\phi)^2] = 0 \quad (variation\ w.r.t.\ \phi) \tag{1.5}$$

Now let us trace $eq^n(1.4)$ and subtracting from it twice of $eq^n(1.5)$ we get :-

$$2(\nabla\phi)^2 - \nabla^2\phi - 2\lambda^2 - 2(\nabla\phi)^2 + 2\nabla^2\phi + 2\lambda^2 + \frac{R}{2} = 0$$
$$\Rightarrow \nabla^2\phi + \frac{R}{2} = 0 \tag{1.6}$$

Substituting $eq^n(1.6)$ in $eq^n(1.5)$ we get :-

$$(\nabla\phi)^2 - \lambda^2 + \frac{R}{4} = 0 \tag{1.7}$$

Now let us consider the vacuum in this model given by $R = 0$ (one should not confuse this vacuum as the vacuum solution obtained in the case of $T_{\mu\nu} = 0$ which is already the case; rather this vacuum is analogically to the Minkowski vacuum for an arbitrary (3+1) dimensional metric model). Thus, the equations describing the vacuum are (from $eq^n(1.6)$ and $eq^n(1.7)$) :-

$$\nabla^2\phi = 0 \tag{1.8}$$
$$(\nabla\phi)^2 = \lambda^2 \tag{1.9}$$

We will describe the vacuum after we obtain the general solution. Now, since every (1+1) dimensional metric is conformally flat so we write :-

$$ds^2 = -e^{2\rho}dudv \tag{1.10}$$

where, we will call new coordinates (u, v) as the *"exponential null coordinates"*. They are related to the usual null coordinates as :-

$$\lambda u = -e^{-\lambda\tilde{u}} = -e^{-\lambda(x^0-x^1)} \tag{1.11}$$
$$\lambda v = e^{\lambda\tilde{v}} = e^{\lambda(x^0+x^1)} \tag{1.12}$$

We will show that metric in $eq^n(1.10)$ is conformally flat indeed. Using $eq^n(1.11)$ and $eq^n(1.12)$ we have :-

$$\begin{aligned} ds^2 &= -e^{2\rho}dudv = e^{2\rho}\lambda^2 uvd\tilde{u}d\tilde{v} \\ &= e^{2\rho}(-e^{\lambda(\tilde{v}-\tilde{u})})d\tilde{u}d\tilde{v} = e^{2\rho}e^{2\lambda x^1}(-d\tilde{u}d\tilde{v}) \\ &= -e^{2\tilde{\rho}}d\tilde{u}d\tilde{v} \end{aligned} \tag{1.13}$$

where, $\tilde{\rho} = \rho + \lambda x^1$. Indeed, metric in $eq^n(1.13)$ is conformally flat.

In the (u, v) coordinates we have :-

$$g_{uv} = g_{vu} = -\frac{1}{2}e^{2\rho}$$
$$g_{uu} = g_{vv} = 0$$
$$\Gamma^u_{uu} = 2\partial_u\rho$$
$$\Gamma^v_{vv} = 2\partial_v\rho$$

Using the above equations one can compute the Ricci scalar curvature to be :-

$$\begin{aligned} R &= g^{\mu\nu}[\Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}] \\ &= -g^{uv}[\Gamma^u_{uu,v} + \Gamma^v_{vv,u}] \end{aligned}$$
$$\Rightarrow R = 8e^{-2\rho}\partial_v\partial_u\rho \tag{1.14}$$

Now let us compute the equations of motion in the (u, v) coordinates :-

$$e^{-2(\phi+\rho)}[-4\partial_v\partial_u\phi + 4\partial_v\phi\partial_u\phi + 2\partial_v\partial_u\rho + \lambda^2e^{2\rho}] = 0 \quad (\textit{variation w.r.t. } \phi) \tag{1.15}$$
$$e^{-2(\phi+\rho)}[2\partial_v\partial_u\phi - 4\partial_u\phi\partial_v\phi - \lambda^2e^{2\rho}] = 0 \quad (\textit{trace of variation w.r.t. } g_{\mu\nu}) \tag{1.16}$$

Now adding the above two equations we get :-

$$\partial_v\partial_u(\rho - \phi) = 0 \tag{1.17}$$

Now consider $eq^n(1.4)$ with $\mu = \nu = u$ first, and then with $\mu = \nu = v$ to get :-

$$2e^{-2\phi}[\nabla_u\nabla_u\phi] = 0$$
$$\Rightarrow e^{-2\phi}[4\partial_u\phi\partial_u\rho - 2\partial_u^2\phi] = 0 \tag{1.18}$$
$$2e^{-2\phi}[\nabla_v\nabla_v\phi] = 0$$
$$\Rightarrow e^{-2\phi}[4\partial_v\phi\partial_v\rho - 2\partial_v^2\phi] = 0 \tag{1.19}$$

$Eq^n(1.18)$ and $eq^n(1.19)$ will be called as constraints.

1.1.3 Solution to the equations of motion

General solution of $eq^n(1.17)$ is :-

$$\rho = \phi + f_v(v) + f_u(u)$$

But we can perform a coordinate transformation, $(u, v) \mapsto (\bar{u}, \bar{v})$ preserving the conformal gauge $eq^n(1.10)$ and then this can be used to set $f_u(u) = f_v(v) = 0$, [1]. Thus we get :-

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$$\rho = \phi \tag{1.20}$$

With $\phi = \rho$ the constraints become :-

$$\partial_u^2(e^{-2\rho}) = e^{-2\rho}[4(\partial_u\rho)^2 - 2\partial_u^2\rho] = 0 \quad (by\ Eq^n(1.18))$$
$$\partial_v^2(e^{-2\rho}) = e^{-2\rho}[4(\partial_v\rho)^2 - 2\partial_v^2\rho] = 0 \quad (by\ Eq^n(1.19))$$

With $\phi = \rho$ in $eq^n(1.15)$ we get :-

$$4\partial_v\rho\partial_u\rho - 2\partial_v\partial_u\rho + \lambda^2 e^{2\rho} = 0$$
$$Now,\ \partial_v\partial_u(e^{-2\rho}) = e^{-2\rho}[4\partial_v\rho\partial_u\rho - \partial_u\partial_v\rho]$$
$$= e^{-2\rho}(-\lambda^2 e^{2\rho}) = -\lambda^2 \quad (using\ above\ equation)$$

Thus, the equation of motion and the constraints in the (u, v) coordinates become :-

$$\partial_v\partial_u(e^{-2\rho}) = -\lambda^2 \tag{1.21}$$
$$\partial_u^2(e^{-2\rho}) = \partial_v^2(e^{-2\rho}) = 0 \tag{1.22}$$

The general solution to the above equations is obtained as :-

$$e^{-2\phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2(u - u_0)(v - v_0)$$

where, M,u_0 and v_0 are integration constants and by shifting u and v, we can set $u_0 = v_0 = 0$. Thus,

$$e^{-2\phi} = e^{-2\rho} = \frac{M}{\lambda} - \lambda^2 uv \tag{1.23}$$

1.1.4 Justification of the solution as eternal black hole solution

The Ricci curvature scalar can be computed as :-

$$R = 8e^{-2\rho}\partial_v\partial_u\rho = 8\left[\frac{M}{\lambda} - \lambda^2 uv\right]\frac{\lambda^2}{2}\left[\frac{\frac{M}{\lambda} - \lambda^2 uv - v(-\lambda^2 u)}{\left(\frac{M}{\lambda} - \lambda^2 uv\right)^2}\right]$$
$$= \frac{4M\lambda}{\frac{M}{\lambda} - \lambda^2 uv} \tag{1.24}$$

Also, the metric is :-

$$ds_D^2 = -\left(\frac{M}{\lambda} - \lambda^2 uv\right)^{-1} dudv \tag{1.25}$$

From the value of the scalar curvature one can see that it blows up at $uv = \frac{M}{\lambda^3}$ which is hence indeed a true curvature singularity. Now, we will show that $uv = 0$ are null hypersurfaces and using a trapped points arguments we will justify that $uv = 0$ are horizons and $u = 0$ will describe the future event horizon (future EH).

So, let us compute the normal vector fields for the metric. Let M be the manifold we are working on, S be the family of hypersurfaces given by, $uv = const.$, to which the normal vector fields are being computed and let $\tilde{f}$ be a smooth function on the manifold, [2], then :-

$$l = \tilde{f}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial}{\partial x^\nu} = \tilde{f}g^{uv}v\frac{\partial}{\partial v} + \tilde{f}g^{vu}u\frac{\partial}{\partial u}$$
$$= -2\tilde{f}e^{-2\rho}\left[v\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\right]$$

Then,

$$l^u = -2\tilde{f}e^{-2\rho}u \ ; \ l^v = -2\tilde{f}e^{-2\rho}v$$
$$l^\mu l_\mu = -4\tilde{f}^2 e^{-2\rho}uv$$

Thus, $|l^\mu l_\mu|_{uv=0} = 0$ implying that $uv = 0$ are null hypersurfaces.

Now let us digress a bit and understand the concepts of trapped surfaces and how an argument from this will justify that $u = 0$ describe the future event horizon. Let us start by defining what are trapped surfaces, [3].

Definition 1.1.1. Trapped surfaces are spacelike or null surfaces whose area tends to decrease along any possible future like direction. In terms of null geodesics these are regions where both the family of null geodesics (i.e., incoming-$\tilde{u}$ and outgoing-$\tilde{v}$) are converging at all points in that region. This implies that the area is decreasing along any possible future like direction.

Recall the (3+1) dimensional metric $ds^2_{(4)}$ and note the are of the 2-spheres are :-

$$A = \frac{4\pi e^{-2\phi(\tilde{u},\tilde{v})}}{\lambda^2} \tag{1.26}$$

Following the definition of trapped surfaces above we have in that region :-

$$\begin{aligned} &\partial_{\tilde{u}} A < 0 \ \ and \ \ \partial_{\tilde{v}} A < 0 \\ \Rightarrow\ &\partial_{\tilde{u},\tilde{v}} A < 0 \\ \Rightarrow\ &\partial_{\tilde{u},\tilde{v}}(e^{-2\phi}) < 0 \\ \Rightarrow\ &\partial_{\tilde{u},\tilde{v}}\phi > 0 \end{aligned} \tag{1.27}$$

Because 2D Dilaton gravity model is obtained by spherical symmetric dimensional reduction of the (3+1) dimensional spacetime the 2-spheres in the above metric will be mapped to points in the dilaton model and hence instead of trapped surfaces we will be trapped points where $\partial_{\tilde{u},\tilde{v}}\phi > 0$. Now, we will define what is known as an apparent horizon (AH).

Definition 1.1.2. Apparent horizon is defined to be the boundary of the trapped region.

Note that, inside the AH $\partial_{\tilde{u},\tilde{v}}\phi > 0$. Now, $\tilde{u}$'s converge and $\tilde{v}$'s diverge at any arbitrary point other than inside the AH. So, outside the AH $\tilde{v}$'s diverge $[i.e., \partial_{\tilde{v}}(e^{-2\phi}) > 0 \ \Rightarrow \ \partial_{\tilde{v}}\phi < 0]$ and inside the AH $\tilde{v}$'s converge. Hence on the AH :-

$$\partial_{\tilde{v}}\phi = 0 \tag{1.28}$$

Now since, $\partial_{\tilde{v}} = \lambda v \partial_v$, hence $eq^n(1.28)$ implies :-

$$\partial_v \phi = 0 \tag{1.29}$$

Since we are at this point interested in eternal black hole solutions where $AH = EH$, thus for the metric, ds^2_D, we get the EH as :-

$$\begin{aligned} &\partial_v\phi = \frac{1}{2}\frac{\lambda^2 u}{\frac{M}{\lambda} - \lambda^2 uv} = 0 \\ \Rightarrow\ &u = 0 \end{aligned} \tag{1.30}$$

Thus, the above equation shows that indeed, $uv = 0$ describe horizons and, $u = 0$ describe the future event horizon. This shows that the model up until now describes eternal black hole solutions with M being the mass of the black hole, true curvature singularity being at $uv = \frac{M}{\lambda^3}$ and the future event horizon being at $u = 0$.

1.1.5 Revisiting the vacuum

Let us study the vacuum solution which is described by $eq^n(1.8)$ and $eq^n(1.9)$. So, for this we will move to the $(\tilde{u}, \tilde{v})$ coordinates.

Then,

$$\begin{aligned} \phi &= -\frac{1}{2}ln\left(\frac{M}{\lambda} - \lambda^2 uv\right) = ln\left(\frac{M}{\lambda} + e^{2\lambda x^1}\right)^{-1/2} \\ &= ln\frac{e^{-\lambda x^1}}{\left(1 + \frac{M}{\lambda}e^{-2\lambda x^1}\right)^{1/2}} = -\lambda x^1 + ln\left(1 + \frac{M}{\lambda}e^{-2\lambda x^1}\right)^{-1/2} \\ \Rightarrow \lim_{x^1\to\infty} \phi &= -\lambda x^1 + ln\left(1 - \frac{M}{2\lambda}e^{-2\lambda x^1}\right) =_{x^1\to\infty} \left(-\lambda x^1 - \frac{M}{2\lambda}e^{-2\lambda x^1}\right) \end{aligned} \tag{1.31}$$

Thus, in the limit of $x^1 \to \infty$ (from $eq^n(1.31)$) :-

$$\phi \to -\lambda x^1 \ \ (upto\ leading\ order\ in\ x^1) \tag{1.32}$$

$$e^{-2\phi} = e^{-2\rho} = -\lambda^2 uv \tag{1.33}$$

$$R = 8e^{-2\rho}\partial_v\partial_u\rho = 8(-\lambda^2 uv)\partial_v\partial_u\left(-\frac{1}{2}ln(-\lambda^2 uv)\right) = 0 \tag{1.34}$$

Also,

$$ds_D^2 = \frac{dudv}{\lambda^2 uv} = -d\tilde{u}d\tilde{v} \tag{1.35}$$

which is nothing but the flat metric in (1+1) dimensions. Thus, the above equations show that the metric, ds_D^2 is asymptotically flat and the vacuum satisfying $eq^n(1.8)$ and $eq^n(1.9)$ is given by :-

$$\phi = -\lambda x^1 \tag{1.36}$$

The above vacuum is conventionally known as the *"Linear Dilaton Vacuum"*.

We will like to make a comment here that in (1+1) dimensions the *Riemann* tensor has $\frac{n^2(n^2-1)}{12} = 1\ (for\ n=2)$ independent component. One can even show that, $R_{\lambda\mu\nu\kappa} = \frac{R}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})$. Hence, the independent component of $R_{\lambda\mu\nu\kappa}$ is R and $R=0$ implies the vanishing of all components of $R_{\lambda\mu\nu\kappa}$ implying the corresponding spacetime being flat.

1.1.6 *Carter-Penrose* Diagrams

Let us draw *Carter-Penrose* diagrams for the vacuum and for the eternal black hole solution. For all diagrams we will consider a coordinate transformation of the form :-

$$\tilde{u} = \tan\hat{u} \ \ , \ \ \tilde{v} = \tan\hat{v} \tag{1.37}$$

$$\tau = \hat{u} + \hat{v} \ \ , \ \ \chi = \hat{v} - \hat{u} \tag{1.38}$$

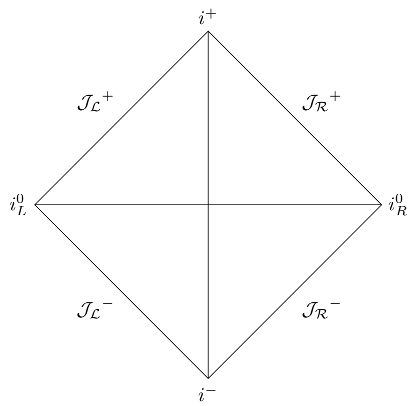


Figure 1.1: *Carter-Penrose* diagram for the linear dilaton vacuum solution.

Observe the similarity of the above diagram with that of the *Carter-Penrose* diagram of Minkowski spacetime but in the latter case $r \geq 0$ hence we only have the right half but here its a complete diamond since here $x^1 \in (-\infty, \infty)$. Now, let us plot the *Kruskal* diagram for the eternal black hole solution.

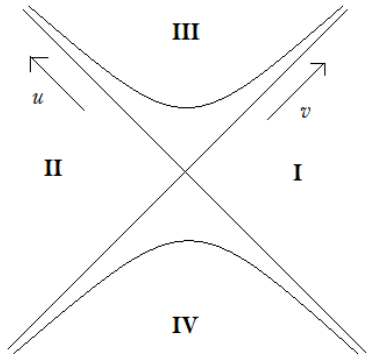


Figure 1.2: *Kruskal* diagram for the eternal black hole solution. $Hyperbolas \ : - \ uv = \frac{M}{\lambda^3}$.

Observe the similarity of the above diagram with that of the *Kruskal* diagram of Schwarzschild spacetime. Now, let us construct the *Carter-Penrose* of the eternal black hole solution from the *Kruskal* diagram above and with the help of the coordinate transformations given in $eq^n(1.37)$ and $eq^n(1.38)$.

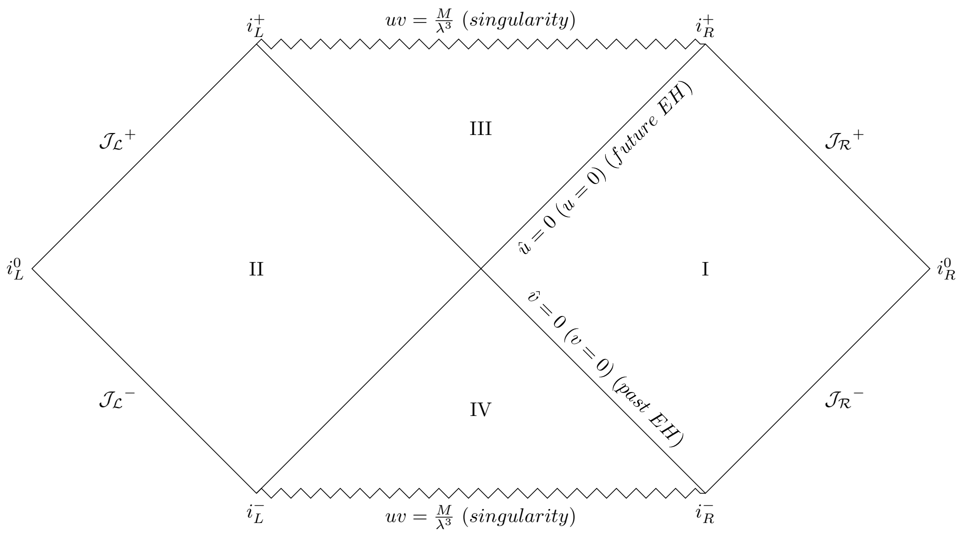


Figure 1.3: *Carter-Penrose* diagram for the eternal black hole solution.

Again, observe the similarity of the above diagram with that of the *Carter-Penrose* diagram of Schwarzschild spacetime, but in the above case as explained in the *Carter-Penrose* diagram of the linear dilaton vacuum case, both the left and the right halves are present.

Note that, $g_s = e^{\phi}$ is a natural coupling constant of the model and in the linear dilaton vacuum case since $\phi = -\lambda x^1$ we have,

$$e^{\phi} >> 1 \quad \Rightarrow \quad x^1 \rightarrow -\infty \quad (strong\ coupling)$$
$$e^{\phi} << 1 \quad \Rightarrow \quad x^1 \rightarrow \infty \quad (weak\ coupling)$$

Furthermore, at the future EH, i.e., $u = 0$, we have :-

$$\begin{aligned} g_s &= \left(\frac{M}{\lambda} - \lambda^2 uv \right)^{-1/2} \\ &= \left(\frac{\lambda}{M} \right)^{1/2} \rightarrow 0 \ \ if \ \ M >> \lambda \end{aligned} \tag{1.39}$$

Thus, in region I of Fig-(1.3) we are always in the weak coupling region for sufficiently massive black holes with $M >> \lambda$. Hence, λ sets the energy scale of the model. From now our main focus of interest will be region I of Fig-(1.3) unless otherwise specified.

In the next section we will couple matter fields to the model. That will enable us to study black hole formation in 2D dilaton gravity model.

1.2 Coupling to matter fields

1.2.1 Setting up the Lagrangian

Consider the matter action as follows, [1] :-

$$S_M = -\frac{1}{4\pi}\int d^2x\sqrt{-g}(\nabla f)^2 \tag{1.40}$$

where, f is the massless scalar field. Now, the complete action is $S_{D+M} = S_D + S_M$:-

$$S_{D+M} = \frac{1}{2\pi}\int d^2x\sqrt{-g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}(\nabla f)^2\right] \tag{1.41}$$

1.2.2 Equations of motion and the Stress tensor

Varying the above action S_{D+M} gives the equations of motion as follows :-

$$4e^{-2\phi}[\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}((\nabla\phi)^2 - \nabla^2\phi - \lambda^2)] = \nabla_\mu f\nabla_\nu f - \frac{1}{2}g_{\mu\nu}(\nabla f)^2 \quad (variation\ w.r.t.\ g_{\mu\nu}) \tag{1.42}$$

$$2e^{-2\phi}[R + 4\lambda^2 + 4\nabla^2\phi - 4(\nabla\phi)^2] = 0 \quad (variation\ w.r.t.\ \phi) \tag{1.43}$$

$$\nabla^2 f = 0 \quad (variation\ w.r.t.\ f) \tag{1.44}$$

$Eq^n(1.44)$ in (u, v) coordinates is :-

$$\partial_v\partial_u f = 0 \tag{1.45}$$

whose general solution is :-

$$f(u, v) = f_{in}(v) + f_{out}(u) \tag{1.46}$$

Now let us compute the stress tensor for the matter field lagrangian, $\mathcal{L} = \frac{1}{2}g^{\mu\nu}\nabla_\mu f\nabla_\nu f$:-

$$\begin{aligned} T_{\mu\nu} &= 2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}g_{\mu\nu} \\ &= \nabla_\mu f\nabla_\nu f - \frac{1}{2}g_{\mu\nu}(\nabla f)^2 \qquad (1.47) \\ \Rightarrow\ T_{vv} &= (\partial_v f)^2 = (\partial_v f_{in})^2 \qquad (1.48) \end{aligned}$$

Note that, the trace of $eq^n(1.42)$ remains same as before, thus, we again get (after writing the ϕ and $g_{\mu\nu}$ equations of motion in (u, v) coordinates and adding them), $\partial_v\partial_u(\rho - \phi) = 0$; and hence by previous argument we can set $\rho = \phi$. Furthermore,

$$\nabla_v\nabla_v\phi = \nabla_v f\nabla_v f = (\partial_v f_{in})^2$$

Thus, the g_{vv} constraint becomes (the g_{uu} constraint still remains same) :-

$$e^{-2\phi}[4\partial_v\phi\partial_v\rho - \partial_v^2\phi] = -\frac{1}{2}(\partial_v f_{in})^2 \tag{1.49}$$

With $\phi = \rho$ we have for the g_{vv} constraint :-

$$\partial_v^2(e^{-2\rho}) = -\frac{1}{2}(\partial_v f_{in})^2 \tag{1.50}$$

The other equation of motion $eq^n(1.21)$ still holds as tracing the eq^n obtained after varying S_{D+M} still is same as $eq^n(1.16)$.

1.2.3 Choosing a particular $f - wave$ and the solution

Let us choose a δ $function$ $f - wave$ as :-

$$T_{vv}(v) = 2a\delta(v - v_0) \tag{1.51}$$

$$\Rightarrow \frac{1}{2}(\partial_v f_{in})^2 = a\delta(v - v_0) \tag{1.52}$$

for some $v_0 \in [0, \infty)$ since $f - waves$ are sent in from $\mathcal{J}_{\mathcal{R}}^-$.

Thus, for $v < v_0$ we have the linear dilaton vacuum and for $v > v_0$ we have the 2D dilatonic black hole solution, hence upon integrating w.r.t v we get :-

$$e^{-2\phi} = e^{-2\rho} = -a(v - v_0)\Theta(v - v_0) - \lambda^2 uv \tag{1.53}$$

One can identify the mass of the black hole formed in the above case as $av_0\lambda$ after sifting u by a/λ^2.

1.2.4 *Carter-Penrose* Diagram

The *Carter-Penrose* diagram for the 2D dilatonic black hole formation process is given below.

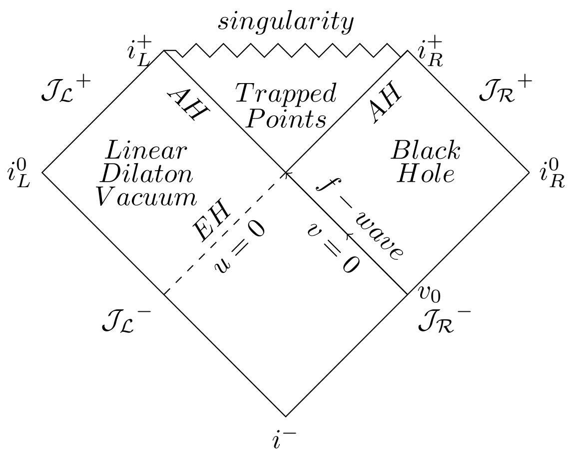


Figure 1.4: *Carter-Penrose* diagram for the 2D dilatonic black hole formation.

Note that, in the above diagram one can notice the similarity between it and with the diagram for the linear dilaton vacuum for $v < v_0$ and with the diagram for the 2D eternal dilatonic black hole for $v > v_0$. That is exactly how such a *Carter-Penrose* diagram for black hole formation is constructed.

In the next chapter we will be studying the 2D dilaton gravity model with a dynamical boundary. One motivation for such a study one can think of is looking at the similarity of Fig-(1.4) with that of the *Carter-Penrose* diagram for spherically symmetry matter collapse. The similarity is very much striking if in Fig-(1.4) there is a boundary separating the right and the left halves. Indeed in the next chapter we will see that the *Carter-Penrose* diagram for black hole formation in such a setting matches with that of the *Carter-Penrose* diagram for spherically symmetry matter collapse.

Chapter 2

2D Dilaton Gravity with a boundary

In this chapter we will study the 2D Dilaton gravity model with a dynamical boundary. As before we will send incoming matter waves ($f - waves$) from $\mathcal{J}_{\mathcal{R}}^{-}$ towards the boundary and we will see that $\exists$ a critical value of the energy of the incoming waves below which the waves are perfectly reflected and go off to $\mathcal{J}_{\mathcal{R}}^{+}$ and beyond which a black hole formation occurs at a point on the boundary and some waves are sucked in and some waves are reflected to $\mathcal{J}_{\mathcal{R}}^{+}$.

2.1 Dynamical Boundary term

Consider the following action, [4] :-

$$\begin{aligned} S_B = &\int_{\phi<\phi_0} d^2x\sqrt{-g}\left[e^{-2\phi}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}(\nabla f)^2\right] \\ + &\underbrace{\int_{\phi=\phi_0} d\tau e^{-2\phi}[2K + 4\lambda]}_{S_b\ :=\ boundary\ term\ in\ the\ action} \end{aligned} \tag{2.1}$$

where, τ is the proper time on the boundary, $K = g^{\mu\nu}\nabla_\mu n_\nu$ is the extrinsic curvature of the boundary $\phi = \phi_0$ and $n_\mu\ \alpha\ \nabla_\mu\phi$ is the normal vector of the boundary. The first part in S_B is same as S_{D+M} in $eq^n(1.41)$.

Now, let us define :-

$$\tilde{\phi} = \phi - \phi_0 \tag{2.2}$$

$$\tilde{f} = e^{\phi_0} f \tag{2.3}$$

Then,

$$S_B = e^{-2\phi_0}\left[\int_{\tilde{\phi}<0} d^2x\sqrt{-g}\left[e^{-2\tilde{\phi}}(R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2}(\nabla\tilde{f})^2\right] + \int_{\tilde{\phi}=0} d\tau e^{-2\tilde{\phi}}[2K + 4\lambda]\right] \tag{2.4}$$

$$= \tilde{S_B} e^{-2\phi_0} \tag{2.5}$$

$$\Rightarrow\ S_B = \frac{\tilde{S_B}}{e^{2\phi_0}} \tag{2.6}$$

$Eq^n(2.6)$ shows that $e^{2\phi_0}$ plays the role of a *Planck* constant in the model and hence, S_B remains classical for $e^{2\phi_0} << 1$ which means $\phi_0 << -1$.

2.1.1 Equations of motion

Now let us vary S_B to obtain the equations of motion. For the bulk part, i.e.; for $\phi < \phi_0$, we ignore the boundary term S_b in the action and varying the remaining of S_B, we get the equations of motion are same as $Eq^n(1.42)$, $Eq^n(1.43)$ and $Eq^n(1.44)$. Moving onto deriving the boundary equations of motion, we get :-

$$n^\mu \nabla_\mu \phi = \lambda \quad , \quad n^\mu \nabla_\mu f = 0 \quad at \quad \phi = \phi_0 \tag{2.7}$$

Note that on the boundary at $\phi = \phi_0$:-

$$\begin{aligned} Now,\ & n^\mu = \alpha \nabla^\mu \phi \\ & \Rightarrow n^\mu n_\mu = \alpha^2 (\nabla\phi)^2 = 1 \\ Then,\ & n^\mu \nabla_\mu \phi = \lambda \ \Rightarrow \ \alpha(\nabla\phi)^2 = \lambda \\ & \Rightarrow \alpha^2 (\nabla\phi)^4 = \lambda^2 \ \Rightarrow \ n^\mu n_\mu (\nabla\phi)^2 = \lambda^2 \\ & \Rightarrow (\nabla\phi)^2 = \lambda^2 \end{aligned} \tag{2.8}$$

---

where, we have assumed that the boundary is not nulllike and hence $n^\mu n_\mu \neq 0$. Then, from $eq^n(2.8)$ it is clear that (using $eq^n(1.9)$), $R = 0$ on the boundary implying the spacetime is flat on the boundary.

Now, moving onto the conformal gauge $g_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$, let us compute the stress tensor for the model as before :-

$$T_{vv} = (\partial_v f)^2 = (\partial_v f_{in})^2 \tag{2.9}$$

$$T_{uu} = (\partial_u f)^2 = (\partial_u f_{out})^2 \tag{2.10}$$

where, $f(u, v) = f_{in}(v) + f_{out}(u)$ since here also f satisfies $\nabla^2 f = 0$.

We will make a comment hear regarding the above stress tensor components that in the previous case with the action S_{D+M}, we had $T_{uu} = 0$ but here $T_{uu} \neq 0$ as here we have a boundary in this case and incoming waves from $\mathcal{J}^-_{\mathcal{R}}$ will be reflected off the boundary to $\mathcal{J}^+_{\mathcal{R}}$ under some conditions.

As before, in the (u, v) coordinates, with $\phi = \rho$, we have the equations of motion as :-

$$\partial_v^2 (e^{-2\rho}) = -\frac{1}{2}(\partial_v f_{in})^2 \tag{2.11}$$

$$\partial_u^2 (e^{-2\rho}) = -\frac{1}{2}(\partial_u f_{out})^2 \tag{2.12}$$

$$\partial_v \partial_u (e^{-2\rho}) = -\lambda^2 \tag{2.13}$$

2.1.2 The Solution to the equations of motion

Then, the solution is (upto shifting in u and v as before, i.e.; setting integration constants $u_0 = v_0 = 0$) :-

$$e^{-2\phi} = e^{-2\rho} = \frac{M_-}{\lambda} - \lambda^2 uv + g(v) + h(u) \tag{2.14}$$

where,

$$g(v) = -\frac{1}{2}\int_0^v dv' \int_{v'}^\infty dv'' T_{vv}(v'') \tag{2.15}$$

$$h(u) = -\frac{1}{2}\int_{-\infty}^u du' \int_{-\infty}^{u'} du'' T_{uu}(u'') \tag{2.16}$$

Now we will show that M_- represents the mass of a white hole and we will hence neglect it since we will demand that in the past null infinity $\mathcal{J}^-_{\mathcal{R}}$ (which is the asymptotic region where incoming waves are sent from) is flat. From $eq^n(1.14)$ and $eq^n(1.24)$ we know that (we will further assume that no $f - waves$ are present) :-

$$R = \frac{4M_-\lambda}{\frac{M_-}{\lambda} - \lambda^2 uv} = 4M_-\lambda e^{2\phi} \quad (using\ eq^n(2.14)) \tag{2.17}$$

Now at $\mathcal{J}^-_{\mathcal{R}}$ (where, $u \to -\infty,\ v \to 0$) with $\phi = const.$:-

$$R = 4M_-\lambda e^{2\phi} \neq 0 \ \Leftrightarrow \ M_- \neq 0 \tag{2.18}$$

Thus, since M_- exists at $\mathcal{J}^-_{\mathcal{R}}$ it represents the mass of a white hole and since we will demand starting from a flat spacetime, we set :-

$$M_- = 0 \tag{2.19}$$

Hence, the solution now becomes :-

$$e^{-2\phi} = e^{-2\rho} = -\lambda^2 uv + g(v) + h(u) \tag{2.20}$$

2.2 Boundary Properties

Let us move onto studying the boundary properties. For this we will describe the boundary trajectory as :-

$$u = U(v) \quad such\ that, \quad \phi(U(v), v) = \phi_0 \tag{2.21}$$

Now, since $\phi = \phi_0(const.)$ on the boundary we have from $eq^n(2.20)$:-

$$\frac{d}{dv}(e^{-2\phi})\Big|_{\phi=\phi_0} = 0 = U^{'}[\partial_u h - \lambda^2 v] + \partial_v g - \lambda^2 U \tag{2.22}$$

Let us now digress a bit and compute the normal vector fields for the boundary $u = U(v)$. For this we define :-

$$S(u, v) = u - U(v) = 0 \tag{2.23}$$

The above equation describe the hypersurface concerning the boundary. Then,

$$\begin{aligned} &\partial_v S = -U^{'} \\ &\partial_u S = 1 \\ &N^2(= norm^2) = g^{uv}\partial_u S \partial_v S + g^{vu}\partial_v S \partial_u S = 4e^{-2\phi_0}U^{'} \end{aligned} \tag{2.24}$$

Thus from $eq^n(2.24)$ one can see that :-

$$\begin{aligned} N^2 > 0 \;&\Leftrightarrow\; U^{'} > 0 \;\; boundary\ is\ timelike \\ N^2 = 0 \;&\Leftrightarrow\; U^{'} = 0 \;\; boundary\ is\ nulllike \\ N^2 < 0 \;&\Leftrightarrow\; U^{'} < 0 \;\; boundary\ is\ spacelike \end{aligned} \tag{2.25}$$

Then moving onto computing the normal vector fields :-

$$n_u = \frac{\partial_u S}{N} = \frac{1}{2e^{-\phi_0}\sqrt{U'}} \quad , \quad n_v = \frac{\partial_v S}{N} = -\frac{\sqrt{U'}}{2e^{-\phi_0}} \tag{2.26}$$

$$n^u = \sqrt{U'}e^{-\phi_0} \quad , \quad n^v = -\frac{1}{\sqrt{U'}}e^{-\phi_0} \tag{2.27}$$

Now let us do some computations. Differentiating $eq^n(2.20)$ first w.r.t. u while treating v as constant (then vice-versa) and evaluating on the boundary we get :-

$$-\lambda^2 v + \partial_u h = (-2e^{-2\phi})\partial_u\phi\Big|_{\phi=\phi_0} \;\Rightarrow\; \partial_u\phi|_{\phi=\phi_0} = \frac{(-\lambda^2 v + \partial_u h)}{(-2e^{-2\phi_0})} \quad (w.r.t.\ u) \tag{2.28}$$

$$-\lambda^2 u + \partial_v g = (-2e^{-2\phi})\partial_v\phi\Big|_{\phi=\phi_0} \;\Rightarrow\; \partial_v\phi|_{\phi=\phi_0} = \frac{(-\lambda^2 u + \partial_v g)}{(-2e^{-2\phi_0})} \quad (w.r.t.\ v) \tag{2.29}$$

Now from the first equation of $eq^n(2.7)$ we have :-

$$\begin{aligned} &n^u\partial_u\phi + n^v\partial_v\phi = \lambda \\ &e^{-\phi_0}\sqrt{U'}\frac{(-\lambda^2 v + \partial_u h)}{-2e^{-2\phi_0}} - \frac{1}{\sqrt{U'}}e^{-\phi_0}\frac{\partial_v g - \lambda^2 u}{-2e^{-2\phi_0}} = \lambda \quad (using\ eq^n(2.27)\ eq^n(2.28)\ and\ eq^n(2.29)) \\ &\frac{1}{U'}(\partial_v g - \lambda^2 u) - \frac{2\lambda}{e^{\phi_0}\sqrt{U'}} = (\partial_u h - \lambda^2 v) \\ &U^{'}\left[\frac{1}{U'}(\partial_v g - \lambda^2 U) - \frac{2\lambda}{e^{\phi_0}\sqrt{U'}}\right] = \lambda^2 U - \partial_v g \quad (using\ eq^n(2.22)) \end{aligned} \tag{2.30}$$

Thus, from $eq^n(2.30)$ we have for the boundary, $u = U(v)$:-

$$U^{'} = \frac{e^{2\phi_0}}{\lambda^2}(\partial_v g - \lambda^2 U)^2 \tag{2.31}$$

Now moving onto the second equation of $eq^n(2.7)$ we have :-

$$\begin{aligned} &n^u\partial_u f + n^v\partial_v f = 0 \\ &e^{-\phi_0}\sqrt{U'}\partial_u f - \frac{e^{-\phi_0}}{\sqrt{U'}}\partial_v f = 0 \quad (using\ eq^n(2.27)) \\ &\sqrt{U'}\partial_u(f_{out}(u)) = \frac{1}{\sqrt{U'}}\partial_v(f_{in}(v)) \\ &U^{'}\partial_U(f_{out}(U(v))) = \partial_U(f_{in}(v))U^{'} \\ &U^{'}[\partial_U(f_{out}(U(v))) - \partial_U(f_{in}(v))] = 0 \\ &\partial_U(f_{out}(U(v))) - \partial_U(f_{in}(v)) = 0 \\ &[\partial_v(f_{out}(U(v))) - \partial_v(f_{in}(v))]\frac{\partial v}{\partial U} = 0 \quad (say,\ v = V(u),\ i.e.;\ inverting\ u = U(v)) \\ &\partial_v[(f_{out}(U(v))) - (f_{in}(v))] = 0 \\ &f_{out}(U(v)) = f_{in}(v) + const. \end{aligned} \tag{2.32}$$

Matching waves at the boundary, $u = U(v)$ in eq^n(2.32) we have, [4] :-

$$f_{out}(U(v)) = f_{in}(v) \tag{2.33}$$

Eq^n(2.22), eq^n(2.31) and eq^n(2.33) are the equations of motion for the boundary $u = U(v)$. In the next computation we will show that using eq^n(2.31) and eq^n(2.33) we can arrive at eq^n(2.22).

So,

$$\begin{aligned}\frac{d}{dv}\left(\frac{\partial_v g - \lambda^2 U}{U'}\right) &= \lambda^2 e^{-2\phi_0}(\partial_v g - \lambda^2 U)^{-1} \quad (using\ eq^n(2.31)) \\ &= -\frac{\lambda^2 e^{-2\phi_0}}{(\partial_v g - \lambda^2 U)^2}(\partial_v^2 g - \lambda^2 U') = \frac{1}{U'}\left[\lambda^2 U' + \frac{(\partial_v f_{in})^2}{2}\right] \quad (using\ eq^n(2.15)) \\ &= \frac{1}{U'}\left[\lambda^2 U' + \frac{(\partial_u f_{out})^2 U'^2}{2}\right] = \lambda^2 + \frac{(\partial_u f_{out})^2 U'}{2}\end{aligned} \tag{2.34}$$

$$\frac{d}{dv}(\lambda^2 v - \partial_u h) = \lambda^2 - \partial_v \partial_u h = \lambda^2 - U' \partial_u^2 h = \lambda^2 + U' \frac{(\partial_u f_{out})^2}{2} \quad (using\ eq^n(2.16)) \tag{2.35}$$

One can note that eq^n(2.34) and eq^n(2.35) are same. Hence,

$$\frac{d}{dv}\left(\frac{\partial_v g - \lambda^2 U}{U'}\right) = \frac{d}{dv}(\lambda^2 v - \partial_u h) \tag{2.36}$$

which upon integrating is equivalent to eq^n(2.22) upto an additive integration constant v_0 which as before we can set to zero by shifting v.

So, everything is consistent up until now. Let us consider empty space ($T_{vv} = T_{uu} = 0$) for now and see how the boundary equations look like. Then,

$$e^{-2\phi} = e^{-2\rho} = -\lambda^2 uv \quad , \quad f = 0 \tag{2.37}$$

$$U' = e^{2\phi_0}\lambda^2 U^2$$

$$U(v) = -\frac{e^{-2\phi_0}}{\lambda^2 v} \tag{2.38}$$

From eq^n(1.33) one can see that eq^n(2.37) and eq^n(2.38) is indeed the *linear dilaton vacuum*. One can now see that the problem is well-poised and in principle solvable. To see this note that, [4] :-

(1) Initially we are given $f_{in}(v)$ or $T_{vv}(v)$.
(2) Using above inputs one can solve eq^n(2.31) to obtain $U(v)$. Initial conditions for eq^n(2.31) being, at $\mathcal{J}_\mathcal{R}^-$, *i.e.*; $(u \to -\infty,\ v \to 0)$ since $R = 0$, we have the *linear dilaton vacuum* so, for $v \to 0$ we have

$$U \to -\frac{e^{-2\phi_0}}{\lambda^2 v} \tag{2.39}$$

$$e^{-2\phi} = e^{-2\rho} \to -\lambda^2 uv \tag{2.40}$$

$$f \to 0 \tag{2.41}$$

as the initial conditions from eq^n(2.37) and eq^n(2.38).
(3) Then, using eq^n(2.33) one can find $f_{out}(u)$ and that gives f and $T_{uu}(u)$. Now, using eq^n(2.20) we can obtain ϕ and then $g_{\mu\nu}$.

So, in principle the problem is solved. Let us now move onto studying how the black hole formation takes place in this model.

2.3 Black hole formation

Let us define :-

$$\lambda^2 U := \partial_v g - e^{-2\phi_0}\frac{\partial_v \psi(v)}{\psi(v)} \tag{2.42}$$

Then, differentiating eq^n(2.42) w.r.t. v and using eq^n(2.31) we get :-

$$\partial_v^2 \psi(v) = -\frac{e^{2\phi_0}}{2} T_{vv}(v)\psi(v) \tag{2.43}$$

Now let us find a relation between the proper time τ on the boundary and $\psi(v)$. This relation will be useful for later purposes. On the boundary we have :-

$$\begin{aligned} d\tau^2 &= -e^{2\phi_0}dU(v)dv = \frac{e^{4\phi_0}}{\lambda^2}(\partial_v g - \lambda^2 U)^2 dv^2 \\ &= \frac{1}{\lambda^2}\left(\frac{\partial_v \psi}{\psi}\right)^2 dv^2 \quad (using\ eq^n(2.42)) \\ \Rightarrow \frac{d\tau}{dv} &= \frac{1}{\lambda}\frac{\partial_v \psi}{\psi} \\ \Rightarrow \psi(v) &= \psi_0 e^{\lambda\tau(v)} \end{aligned} \tag{2.44}$$

In $eq^n(2.44)$ we will set ψ_0 to be a positive constant since it is the value of ψ when $\tau = 0$. This convention will make sense later as this will be related to the boundary being timelike when the energy of the incoming waves is below a critical value (as one would expect the waves with less energy to reflect off the boundary to future null-infinity).

Also, from $eq^n(2.44)$ we see that at past null-infinity, i.e.; $\mathcal{J}_{\mathcal{R}}^-$ $(u \to -\infty,\ v \to 0)$ and where, $\tau \to -\infty$ we have :-

$$\psi(v) = 0 \quad as \quad v \to 0 \tag{2.45}$$

Now recall, from $eq^n(2.39)$, $eq^n(2.40)$ and $eq^n(2.41)$ as $v \to 0$ we have $\partial_v g = 0$ and hence :-

$$\begin{aligned} \lambda^2 U &= -e^{-2\phi_0}\frac{\partial_v \psi}{\psi} \quad (from\ eq^n(2.42)) \\ \Rightarrow \frac{1}{v} &= \frac{\partial_v \psi}{\psi} \quad (using\ eq^n(2.39)) \\ \Rightarrow \psi(v) &= c_0 v \quad as \quad v \to 0 \end{aligned} \tag{2.46}$$

where, $c_0 > 0$ in accordance with the convention set above by analyzing $eq^n(2.44)$. Now one can see from $eq^n(2.43)$ that if its $RHS \to 0$, i.e.; $T_{vv}(v) \to 0$ (which means for low energetic incoming waves) we have :-

$$\psi(v) \to Cv + D \tag{2.47}$$

with $C > 0$ (as before) and $D = 0$ (from $eq^n(2.46)$).

Note that, since $\partial_v \psi\ \alpha\ -T_{vv}(v)$, the slope, i.e.; C decreases with increase in the incoming energy flux $T_{vv}(v)$. This further implies ψ becomes more and more concave and for a critical incoming energy flux, say $T_{vv} = T_{cr}$, $C = 0$. Beyond this, i.e.; for $T_{vv} > T_{cr}$, $\psi(v)$ will increase first and then will start decreasing. A plot of this is given below.

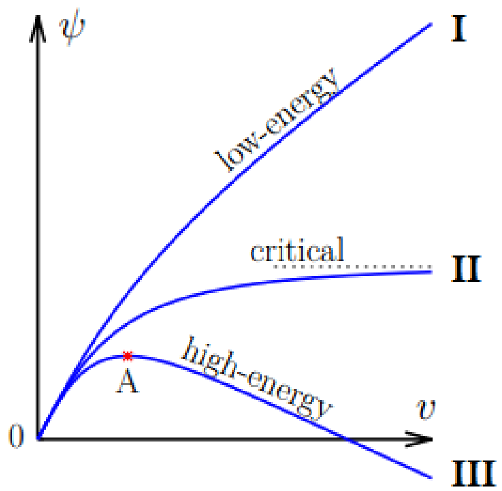


Figure 2.1: Plot of $\psi(v)$ vs v.

Point $A = (u_A, v_A)$ is the point where in the high-energy regime, i.e.; in $T_{vv} > T_{cr}$ regime, the slope C vanishes and after that point with increase in v, $\psi(v)$ starts decreasing.

Now from $eq^n(2.42)$ we have :-

$$\lambda^2 U = \partial_v g \quad at\ A \quad (since,\ \partial_v \psi = 0\ at\ A)$$
$$\Rightarrow\ U^{'}|_A = 0 \quad (using\ eq^n(2.31)) \tag{2.48}$$

So, the boundary is nulllike at A and this also shows that in the critical energy regime (curve II in Fig-(2.1)), the boundary is nulllike at $v \to \infty$. In the curve III of Fig-(2.1), initially, the boundary was timelike as is evident from $eq^n(2.31)$ which shows $U^{'} > 0$, then the boundary becomes nulllike at A and after that with increase in v the boundary is ought to become spacelike. Point A is where the black hole formation takes place. We will elaborate on this below.

Note that (from $eq^n(2.31)$), $U^{''}|_A = 2\frac{e^{2\phi_0}}{\lambda^2}(\partial_v g - \lambda^2 U)(\partial_v{}^2 g - \lambda^2 U^{'})|_A = 0$. With this in mind, let us taylor expand $U(v)$ about $A = (u_A, v_A)$ where, $u_A = U(v_A)$:-

$$U(v) = U(v_A) + U^{'}(v_A)(v - v_A) + \frac{U^{''}(v_A)}{2!}(v - v_A)^2 + \frac{U^{'''}(v_A)}{3!}(v - v_A)^3 + \mathcal{O}(v^4)$$
$$= u_A + d(v - v_A)^3 \tag{2.49}$$
$$So,\ u = u_A + d(v - v_A)^3 \tag{2.50}$$
$$\Rightarrow\ (u - u_A)^2 = d^2(v - v_A)^6$$
$$\Rightarrow\ (u - u_A) = \pm d(v - v_A)^3 \tag{2.51}$$

where, $d = \frac{U^{'''}(v_A)}{3!}$.

So, for $u = U(v)$, i.e.; for $\phi = \phi_0$, we get two intersecting curves at A given by $eq^n(2.51)$. Then,

$$For\ \phi < \phi_0, \quad u = u_A + d(v - v_A)^3$$
$$\Rightarrow\ U^{'} = 3d(v - v_A)^3 > 0 \quad \Rightarrow \quad boundary\ is\ timelike$$
$$For\ \phi > \phi_0, \quad u = u_A - d(v - v_A)^3$$
$$\Rightarrow\ U^{'} = -3d(v - v_A)^3 < 0 \quad \Rightarrow \quad boundary\ is\ spacelike$$

Thus, in the $\phi > \phi_0$ region we obtain a black hole singularity in the high-energy regime. Matter waves in the $u > 0$ region is sucked in limiting the region accessible to outside observer to $u < 0$. Also, as before $u = 0$ is the event horizon.

Now let us plot $U(v)$ vs v keeping in mind that $U\ \alpha\ -\frac{1}{v}$ (in the asymptotic limit) and $U(v)$ goes as cubic near A. Then,

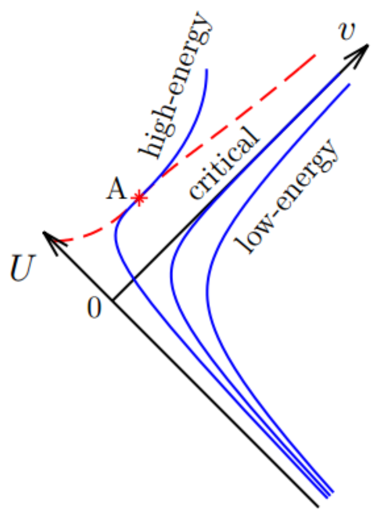
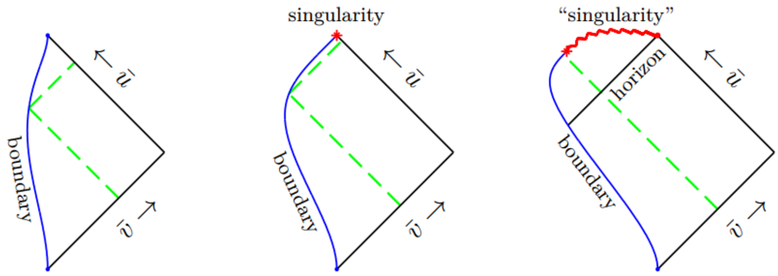


Figure 2.2: Plot of $U(v)$ vs v.

In Fig-(2.2), the dashed-line represents the spacelike curve $u = u_A - d(v - v_A)^3$.

2.3.1 *Carter-Penrose Diagram*

The *Carter-Penrose Diagram* for the black hole formation is given below (with, $u = tan(\bar{u})$ and $v = tan(\bar{v})$).



(a) low-energy (b) critical (c) high-energy

Figure 2.3: *Carter-Penrose* diagram for the black hole formation.

Now suppose we define :-

$$v = \sigma^0 + \sigma^1 \qquad\qquad u = \sigma^0 - \sigma^1 \tag{2.52}$$

then, in the above (σ^0, σ^1) coordinates, $v = 0$ line is the $\sigma^0 = -\sigma^1$ line and Fig-(2.2) becomes :-

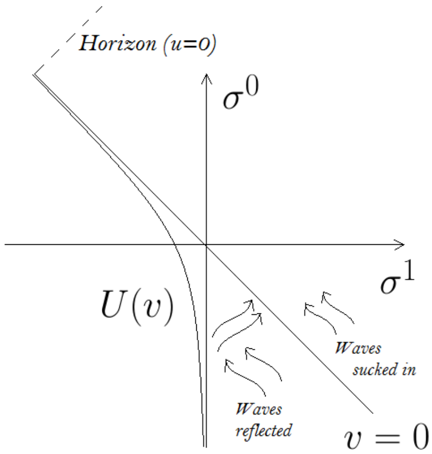


Figure 2.4: Plot of $U(v)$ in (σ^0, σ^1) coordinates. The boundary approaches the nulllike asymptote $v = 0$ and this feature is very much something which is expected in black hole formation. Matter waves sent after $v = 0$ (at later times) will not be reflected and will be sucked in but waves sent before $v = 0$ will reflect off the boundary.

2.3.2 Lower bound on mass of the black hole formed

Let us now estimate the mass of black hole thus formed. By energy conservation we have, [4] :-

$$M_{BH} = \int_0^\infty dv\ \lambda v T_{vv}(v) - \int_{-\infty}^0 du\ \lambda |u| T_{uu}(u) \tag{2.53}$$

$$= \lambda \left[\left| v \int_{v'}^\infty dv\ T_{vv} \right|_0^\infty - \int_0^\infty dv' \left(\int_{v'}^\infty dv\ T_{vv} \right) \right]$$

$$- \lambda \left[\left| |u| \int_{-\infty}^{u'} du\ T_{uu} \right|_{-\infty}^0 + \int_{-\infty}^0 du' \left(\int_{-\infty}^{u'} du\ T_{uu} \right) \right] \quad (integration\ by\ parts)$$

$$= 2\lambda [g(\infty) + h(0)] \quad (using\ eq^n(2.15)\ and\ eq^n(2.16)) \tag{2.54}$$

Recall,

$$e^{-2\phi} = -\lambda^2 uv + g(v) + h(u) \;\Rightarrow\; e^{-2\phi(0,v)} = g(v) + h(0)$$
$$\Rightarrow \lim_{v\to\infty} e^{-2\phi(0,v)} = g(\infty) + h(0) \tag{2.55}$$

Thus,

$$M_{BH} = 2\lambda \lim_{v\to\infty} e^{-2\phi(0,v)} \tag{2.56}$$
$$\Rightarrow \exists\ M_{cr} = 2\lambda e^{-2\phi_0} \quad (as\ \phi < \phi_0 \;\Rightarrow\; e^{-2\phi} > e^{-2\phi_0}) \tag{2.57}$$

Thus, for the black hole, $M_{BH} > M_{cr}$. This critical mass M_{cr} is the lower bound estimate on the mass of the black hole formed.

In the next chapter we will study the phenomenon of Hawking radiation in the 2D dilaton gravity model.

Chapter 3

Hawking Radiation

In this chapter we will study the phenomenon of Hawking radiation from 2D Dilaton gravity models without boundary.

In (1+1) dimensions, there is a relation between the trace anomaly and Hawking radiation, [9]. For a massless scalar field the trace of the stress tensor is zero classically, $T := T^{\mu}_{\mu} = 0$. Quantum mechanically, there is a one-loop anomaly which relates the expectation value of the trace of the stress tensor to the *Ricci* scalar as, [1] :-

$$\langle T \rangle = \frac{c}{24} R \tag{3.1}$$

where, $c = 1$ for massless scalars.

3.1 The outgoing energy flux

Let us calculate the expectation value of the outgoing energy flux.

We have in (u, v) coordinates :-

$$T = T^{\mu}_{\mu} = g^{\mu\nu} T_{\mu\nu} = -4e^{-2\rho} T_{vu}$$

Thus,

$$\begin{aligned} \langle T^{f}_{vu} \rangle &= -\frac{1}{4} e^{2\rho} \langle T \rangle \\ &= -\frac{1}{4} \frac{R}{24} \\ &= -\frac{1}{4} \frac{e^{2\rho}}{24} (8e^{-2\rho} \partial_v \partial_u \rho) \quad (from\ eq^{n}(1.14)) \end{aligned} \tag{3.2}$$

where, the superscript f means final denoting the outgoing flux.

Hence (from $eq^{n}(3.2)$),

$$\langle T^{f}_{vu} \rangle = -\frac{1}{12} \partial_v \partial_u \rho \tag{3.3}$$

Now from energy conservation we have :-

$$\nabla^{\mu} T_{\mu\nu} = g^{\mu\alpha} \nabla_{\alpha} T_{\mu\nu} = 0$$

$$(\mu = v, \nu = v) \ \Rightarrow g^{vu}[\partial_u T_{vv}] = 0 \ \Rightarrow \ \partial_u T_{vv} = 0 \tag{3.4}$$

$$(\mu = u, \nu = v) \ \Rightarrow g^{uv}[\partial_v T_{uv} - \Gamma^{v}_{vv} T_{uv}] = 0 \ \Rightarrow \ \partial_v T_{uv} - \Gamma^{v}_{vv} T_{uv} = 0 \tag{3.5}$$

$$(\mu = u, \nu = u) \ \Rightarrow g^{uv}[\partial_v T_{uu}] = 0 \ \Rightarrow \ \partial_v T_{uu} = 0 \tag{3.6}$$

$$(\mu = v, \nu = u) \ \Rightarrow g^{vu}[\partial_u T_{vu} - \Gamma^{u}_{uu} T_{vu}] = 0 \ \Rightarrow \ \partial_u T_{vu} - \Gamma^{u}_{uu} T_{vu} = 0 \tag{3.7}$$

Thus,

$$Eq^{n}(3.4) \ + \ eq^{n}(3.5) \ \Rightarrow \partial_u T_{vv} + \partial_v T_{uv} - \Gamma^{v}_{vv} T_{uv} = 0 \tag{3.8}$$

$$Eq^{n}(3.6) \ + \ eq^{n}(3.7) \ \Rightarrow \partial_v T_{uu} + \partial_u T_{vu} - \Gamma^{u}_{uu} T_{vu} = 0 \tag{3.9}$$

Upon solving,

$$\partial_u T_{vv} + \partial_v T_{uv} - 2\partial_v \rho T_{uv} = 0$$
$$\partial_v T_{uu} + \partial_u T_{vu} - 2\partial_u \rho T_{vu} = 0$$
$$Then,\ \partial_u \langle T^f_{vv} \rangle = 2\partial_v \rho \left(-\frac{1}{12}\partial_v \partial_u \rho \right) + \frac{1}{12}\partial_u(\partial_v^2 \rho) \quad (from\ eq^n(3.3)) \tag{3.10}$$
$$\partial_v \langle T^f_{uu} \rangle = 2\partial_u \rho \left(-\frac{1}{12}\partial_v \partial_u \rho \right) + \frac{1}{12}\partial_v(\partial_u^2 \rho) \quad (from\ eq^n(3.3)) \tag{3.11}$$

$$Integrating\ eq^n(3.10)\ w.r.t.\ u\ \Rightarrow \langle T^f_{vv} \rangle = -\frac{1}{12}[\partial_v \rho \partial_v \rho - \partial_v^2 \rho + t_v(v)] \tag{3.12}$$
$$Integrating\ eq^n(3.11)\ w.r.t.\ v\ \Rightarrow \langle T^f_{uu} \rangle = -\frac{1}{12}[\partial_u \rho \partial_u \rho - \partial_u^2 \rho + t_u(u)] \tag{3.13}$$

To find out $t_u(u)$ and $t_v(v)$ in $eq^n(3.12)$ and $eq^n(3.13)$, let us introduce new coordinates as :-

$$e^{\lambda y^+} = \lambda v \tag{3.14}$$
$$e^{-\lambda y^-} = -\lambda u - \frac{a}{\lambda} \tag{3.15}$$

where, (y^-, y^+) are coordinates in which the metric ds^2 (concerning black hole formation) is asymptotically flat on $\mathcal{J}^\pm_\mathcal{R}$.

Then, from $eq^n(3.14)$ and $eq^n(3.15)$ we have :-

$$For\ y^+ < y_0^+,\ \ ds^2 = \frac{dudv}{\lambda^2 uv} = \frac{\lambda v \left(-\lambda u - \frac{a}{\lambda}\right) dy^+ dy^-}{\lambda^2 uv} = \frac{(-\lambda^2 uv - av) dy^+ dy^-}{\lambda^2 uv} = -dy^+ dy^- \left[1 + \frac{a}{\lambda^2 u}\right]$$
$$= -dy^+ dy^- \left[1 - \frac{a}{a\lambda e^{-\lambda y^-}}\right] = -\frac{dy^+ dy^-}{\left[1 + \frac{a}{\lambda} e^{\lambda y^-}\right]} \tag{3.16}$$
$$For\ y^+ > y_0^+,\ \ ds^2 = \frac{dudv}{a + \lambda^2 uv} = -\frac{dy^+ dy^-}{\left[1 + \frac{a}{\lambda} e^{\lambda y^- - y^+ + y_0^+}\right]} \tag{3.17}$$

where, $\lambda v_0 = e^{\lambda y_0^+}$.

Thus from $eq^n(3.16)$ and $eq^n(3.17)$ we have :-

$$-2g_{+-} = e^{2\rho} = \begin{cases} \left[1 + \frac{a}{\lambda} e^{\lambda y^-}\right]^{-1}, & if\ \ y^+ < y_0^+ \\ \left[1 + \frac{a}{\lambda} e^{\lambda y^- - y^+ + y_0^+}\right]^{-1}, & if\ \ y^+ > y_0^+ \end{cases} \tag{3.18}$$

In (y^-, y^+) coordinates $eq^n(3.12)$ and $eq^n(3.13)$ become :-

$$\langle T^f_{++} \rangle = -\frac{1}{12}[(\partial_+ \rho)^2 - \partial_+^2 \rho + t_+(y^+)] \tag{3.19}$$
$$\langle T^f_{--} \rangle = -\frac{1}{12}[(\partial_- \rho)^2 - \partial_-^2 \rho + t_-(y^-)] \tag{3.20}$$

To evaluate t_+ and t_- we demand $\langle T^f \rangle$ vanishing in the *linear dilaton vacuum* which means :-

$$\langle T^f_{--} \rangle = \langle T^f_{++} \rangle = \langle T^f_{++} \rangle = 0 \tag{3.21}$$

for $e^{2\rho}$ in the region $y^+ < y_0^+$ which is the *linear dilaton vacuum.* Since, $\langle T^f_{+-} \rangle$ and $\langle T^f_{++} \rangle$ contains differentiation of ρ, w.r.t, y^+ but $e^{2\rho}$ is independent of y^+ they vanish trivially giving $t_+ = 0$.

Now for t_- we obtain :-

$$Since,\ \ \rho = -\frac{1}{2} ln\left(1 + \frac{a}{\lambda} e^{\lambda y^-}\right)$$
$$\Rightarrow \partial_- \rho = -\frac{a e^{\lambda y^-}}{2\left(1 + \frac{a}{\lambda} e^{\lambda y^-}\right)}$$
$$\Rightarrow \partial_-^2 \rho = -\frac{2a\lambda e^{\lambda y^-}}{4\left(1 + \frac{a}{\lambda} e^{\lambda y^-}\right)^2}$$
$$t_- = -\left[\frac{a^2 e^{2\lambda y^-}}{4\left(1 + \frac{a}{\lambda} e^{\lambda y^-}\right)^2} + \frac{2a\lambda e^{\lambda y^-}}{4\left(1 + \frac{a}{\lambda} e^{\lambda y^-}\right)^2}\right] = -\frac{\lambda^2}{4}\left[1 - \frac{1}{1 + \frac{a}{\lambda} e^{\lambda y^-}}\right] \tag{3.22}$$

Now at $\mathcal{J}^+_\mathcal{R}$ where $y^+ \to \infty$, ρ becomes constant as can be seen from value of $e^{2\rho}$ for $y^+ > y^+_0$. Thus at $\mathcal{J}^+_\mathcal{R}$:-

$$\langle T^f_{++} \rangle \to 0 \tag{3.23}$$

$$\langle T^f_{+-} \rangle \to 0 \tag{3.24}$$

$$\langle T^f_{--} \rangle \to \frac{\lambda^2}{48}\left[1 - \frac{1}{1 + \frac{a}{\lambda}e^{\lambda y^-}}\right] \tag{3.25}$$

One can note from eq^n(3.25) that in the far past of $\mathcal{J}^+_\mathcal{R}$, i.e., $y^- \to -\infty$ we have $\langle T^f_{--} \rangle$ vanishing exponentially and while going near the horizon from $\mathcal{J}^+_\mathcal{R}$, i.e., $y^- \to \infty$ we have $\langle T^f_{--} \rangle$ approaching a constant value of $\frac{\lambda^2}{48}$ which we will identify in the upcoming sections as the Hawking radiation. One peculiar feature of these black holes is that the outgoing flux obtained is independent of the mass a of the black hole unlike generic 4D stationary black holes where it depends upon the mass of the black holes.

3.2 In and Out modes

Let us recall the metric for the 2D dilatonic eternal black hole as :-

$$ds^2 = -e^{2\rho}dudv$$

Now let us consider an arbitrary distribution of collapsing matter between v_i and v_f and in this case the solution becomes :-

$$e^{-2\phi} = e^{-2\rho} = -\lambda^2 uv - \int_0^{v_0} dv' \int_{v_0}^{\infty} dv'' T_{vv}(v'') \tag{3.26}$$

where, $v_i \leq v_0 \leq v_f$.

For, $v > v_f$ the last term becomes, [5] :-

$$-\int_0^{v_0} dv' \int_{v_0}^{\infty} dv'' T_{vv}(v'') = \frac{M}{\lambda} - \lambda^2 x^+ \Delta \tag{3.27}$$

where, M and Δ are integration constants. Above is true as for $v < v_i$ the last term above vanishes and we have the *linear dilaton vacuum* and for $v > v_f$ we have the dilatonic black hole.

Then, for $v > v_f$ we have :-

$$ds^2 = -\frac{dudv}{\frac{M}{\lambda} - \lambda^2 v(u + \Delta)} \tag{3.28}$$

Now, the future EH is at $u = -\Delta$. To describe the "in" and "out" modes let us introduce asymptotic coordinates as follows :-

$$For \quad v > v_f, \quad e^{\lambda\sigma^+} = \lambda v, \quad e^{-\lambda\sigma^-} = -\lambda(u + \Delta) \tag{3.29}$$

$$ds^2 = \begin{cases} -\frac{d\sigma^- d\sigma^+}{[1+\Delta\lambda e^{\lambda\sigma^-}]}, & if \quad \sigma^+ < \sigma_i^+ \\ -\frac{d\sigma^- d\sigma^+}{[1+\frac{M}{\lambda}e^{\lambda\sigma^- - \sigma^+}]}, & if \quad \sigma^+ > \sigma_f^+ \end{cases} \tag{3.30}$$

$$For \quad v < v_i, \quad \lambda v = e^{\lambda y^+}, \quad u = -\Delta e^{-\lambda y^-} \tag{3.31}$$

$$ds^2 = -dy^- dy^+ \tag{3.32}$$

where, $\lambda x^+_{i,f} = e^{\lambda\sigma^+_{i,f}}$. From eq^n(3.30), one notes that ds^2 is asymptotically flat at both $\mathcal{J}^+_\mathcal{R}$ where $\sigma^+ \to \infty$ and $\mathcal{J}^-_\mathcal{R}$ where $\sigma^- \to -\infty$. Furthermore, note that in (y^-, y^+) coordinates, $y = 0$ is the future EH.

Now let us consider two asymptotically flat regions $\mathcal{J}^-_\mathcal{L}$ and $\mathcal{J}^+_\mathcal{R}$ which we will call the "in" and the "out" regions respectively. Note that,

$$\nabla^2 f = 0 \quad \Rightarrow \quad \partial_v\partial_u f = 0 \quad \Rightarrow \quad \partial_{\sigma^+}\partial_{\sigma^-} f = 0 \quad \Rightarrow \quad \partial_{y^+}\partial_{y^-} f = 0 \tag{3.33}$$

So, in all the coordinates Laplace equation is satisfied and this enables us to write :-

$$u_\omega = \frac{1}{\sqrt{2\omega}}e^{-i\omega y^-} \quad (in) \quad ; \quad v_\omega = \frac{1}{\sqrt{2\omega}}e^{-i\omega\sigma^-}\Theta(-y^-) \quad (out) \tag{3.34}$$

where, $\omega > 0$. Note that, v_ω has support only outside the horizon $y^- < 0$ and thus this must be complemented, [5] by a set of modes $\hat{v}_\omega$ for the region inside the horizon. Now let us expand f_- in the above bases :-

$$\begin{aligned} f_- &= \int_0^\infty d\omega [a_\omega u_\omega + a_\omega^\dagger u_\omega^*] \quad (in) \\ &= \int_0^\infty d\omega [b_\omega v_\omega + b_\omega^\dagger v_\omega^* + \hat{b}_\omega \hat{v}_\omega + \hat{b}_\omega^\dagger \hat{v}_\omega^*] \quad (out + internal) \end{aligned} \tag{3.35}$$

Now let us define the *Klein-Gordon* inner product as :-

$$(f, g) = -i \int_\Sigma d\Sigma_\mu f \overleftrightarrow{\nabla}_\mu g^* \tag{3.36}$$

where, Σ is an arbitrary *Cauchy* surface. Then, the bases satisfy :-

$$\begin{aligned} (u_\omega, u_{\omega'}) &= (v_\omega, v_{\omega'}) = 2\pi\delta(\omega - \omega') \\ (u_\omega, u_{\omega'}^*) &= (v_\omega, v_{\omega'}^*) = 0 \\ (u_\omega^*, u_{\omega'}^*) &= (v_\omega^*, v_{\omega'}^*) = -2\pi\delta(\omega - \omega') \end{aligned} \tag{3.37}$$

Eq^n(3.35) and eq^n(3.37) together with the canonical commutation relation :-

$$[f_-(x), \partial_0 f_-(x')]_{x^0 = x'^0} = \frac{1}{2}[f(x), \partial_0 f(x')]_{x^0 = x'^0} = i\pi\delta(x^1 - x'^1) \tag{3.38}$$

imply that :-

$$[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega') \quad ; \quad [a_\omega, a_{\omega'}] = 0 = [a_\omega^\dagger, a_{\omega'}^\dagger] \tag{3.39}$$

Finally, the in, out and internal vacua are :-

$$a_\omega|0\rangle_{in} = 0 \quad ; \quad b_\omega|0\rangle_{out} = 0 \quad ; \quad \hat{b}_\omega|0\rangle_{int} = 0 \quad \forall\omega \tag{3.40}$$

Although the in and out regions are flat, their natural timelike coordinates are related in such a way that a field mode which which has positive frequency according to observers in one region becomes a mixture of positive and negative frequencies according to observers in the other region. This mixing can be interpreted as particle creation. To study this let us express one basis in terms of the other bases as :-

$$v_\omega = \int_0^\infty d\omega' [\alpha_{\omega\omega'} u_{\omega'} + \beta_{\omega\omega'} u_{\omega'}^*] \tag{3.41}$$

$$Then, \ \ \alpha_{\omega\omega'} = \frac{1}{2\pi}(v_\omega, u_{\omega'}) \quad ; \quad \beta_{\omega\omega'} = -\frac{1}{2\pi}(v_\omega, u_{\omega'}^*) \tag{3.42}$$

where, $\alpha_{\omega\omega'}$, $\beta_{\omega\omega'}$ are called the *Bogoliubov* coefficients. The *Bogoliubov* coefficients $\hat{\alpha}_{\omega\omega'}$, $\hat{\beta}_{\omega\omega'}$ for the internal modes are defined similarly.

Now equivalence of the expansions in eq^n(3.35) implies :-

$$a_\omega = \int_0^\infty d\omega' [b_{\omega'}\alpha_{\omega\omega'} + b_{\omega'}^\dagger \beta_{\omega\omega'}^* + \hat{b}_{\omega'}\hat{\alpha}_{\omega\omega'} + \hat{b}_{\omega'}^\dagger \hat{\beta}_{\omega\omega'}^*] \tag{3.43}$$

$$b_\omega = \int_0^\infty d\omega' [\alpha_{\omega\omega'}^* a_{\omega'} - \beta_{\omega\omega'}^* a_{\omega'}^\dagger] \tag{3.44}$$

$$\hat{b}_\omega = \int_0^\infty d\omega' [\hat{\alpha}_{\omega\omega'}^* a_{\omega'} - \hat{\beta}_{\omega\omega'}^* a_{\omega'}^\dagger] \tag{3.45}$$

Now if, $\beta_{\omega\omega'} \neq 0$, then the in vacuum is not considered vacuous by the out observer. This means particle creation has occurred. From eq^n(3.44) it follows :-

$${}_{in}\langle 0|N_\omega^{out}|0\rangle_{in} = {}_{in}\langle 0|b_\omega^\dagger b_\omega|0\rangle_{in} = \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2 \tag{3.46}$$

To calculate the *Bogoliubov* coefficients we note the relation between the asymptotic coordinates as :-

$$\sigma^- = -\frac{1}{\lambda} ln[\lambda\Delta(e^{-\lambda y^- - 1})] \tag{3.47}$$

$$Then, \ \ v_\omega = \frac{1}{\sqrt{2\omega}} \exp\left[\frac{i\omega}{\lambda} ln[\lambda\Delta(e^{-\lambda y^-} - 1)]\right]\Theta(-y^-) \tag{3.48}$$

Calculating the inner products in $eq^n(3.42)$ at $\mathcal{J}^-_{\mathcal{L}}$ we get (since, only the in modes contribute at $\mathcal{J}^-_{\mathcal{L}}$ and integrating by parts we can discard the boundary conditions demanding the out modes vanishing at $y^- \to -\infty$ and at $y^- \to 0$, [6] :-

$$\begin{aligned} \alpha_{\omega\omega'} &= -\frac{i}{\pi}\int_{-\infty}^{0} dy^- v_\omega \partial_- u^*_{\omega'} \\ &= \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{0} dy^- \exp\left[\frac{i\omega}{\lambda} ln[\lambda\Delta(e^{-\lambda y^-}-1)] + i\omega' y^-\right] \end{aligned} \tag{3.49}$$

$$\begin{aligned} \beta_{\omega\omega'} &= \frac{i}{\pi}\int_{-\infty}^{0} dy^- v_\omega \partial_- u_{\omega'} \\ &= \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{0} dy^- \exp\left[\frac{i\omega}{\lambda} ln[\lambda\Delta(e^{-\lambda y^-}-1)] - i\omega' y^-\right] \end{aligned} \tag{3.50}$$

Substituting $x = e^{\lambda y^-}$, $\alpha_{\omega\omega'}$ becomes :-

$$\frac{1}{2\pi\lambda}\sqrt{\frac{\omega'}{\omega}}(\lambda\Delta)^{i\omega/\lambda}\int_0^1 dx(1-x)^{i\omega/\lambda}x^{-1+i(\omega'-\omega)/\lambda} \tag{3.51}$$

Then, similarly $\beta_{\omega\omega'}$ can be computed and both altogether in terms of beta functions are given as :-

$$\alpha_{\omega\omega'} = \frac{1}{2\pi\lambda}\sqrt{\frac{\omega'}{\omega - i\epsilon}}(\lambda\Delta)^{i\omega/\lambda}B\left(-\frac{i\omega}{\lambda}+\frac{i\omega'}{\lambda}+\epsilon, 1+\frac{i\omega}{\lambda}\right) \tag{3.52}$$

$$\beta_{\omega\omega'} = \frac{1}{2\pi\lambda}\sqrt{\frac{\omega'}{\omega - i\epsilon}}(\lambda\Delta)^{i\omega/\lambda}B\left(-\frac{i\omega}{\lambda}-\frac{i\omega'}{\lambda}+\epsilon, 1+\frac{i\omega}{\lambda}\right) \tag{3.53}$$

The pole prescriptions are necessary to completely define these quantities, they are chosen so that the expansion $eq^n(3.41)$, and the inverse expansion of u_ω in terms of v_ω, actually hold, [5]. With the pole prescriptions above one can see that the following completeness relation holds :-

$$\int_0^\infty d\omega' [\alpha_{\omega\omega'}\alpha^*_{\omega''\omega'} - \beta_{\omega\omega'}\beta^*_{\omega''\omega'}] = \delta(\omega - \omega'') \tag{3.54}$$

Let us now observe the late time behavior of the *Bogoliubov* coefficients. We are considering late time behavior since at late times the black hole would have settled down and we would have a stationary region in the asymptotic future. Now late time can be characterized in a number of ways. It means $y^- \to 0$ (near the horizon) and equivalently it means $w' \to \infty$ as ω' is the frequency of the incoming wave. All these late time characterization can be better understood by looking at the *Carter-Penrose* diagram for the process in Fig-(3.1). Using $eq^n(3.52)$ and $eq^n(3.53)$:-

$$\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} = \left(\frac{\omega'+\omega}{\omega'-\omega}\right)\frac{\sinh[\pi(\omega'+\omega)/\lambda]}{\sinh[\pi(\omega'-\omega)/\lambda]} \tag{3.55}$$

In the limit $\omega' \to \infty$ we get :-

$$|\beta_{\omega\omega'}| = e^{-\pi\omega/\lambda}|\alpha_{\omega\omega'}| \tag{3.56}$$

Now putting $\omega'' = \omega$ in $eq^n(3.54)$:-

$$\int_0^\infty d\omega' [|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2] = t \tag{3.57}$$

where, t is a large cut-off set equal to $\delta(0)$, [5]. Now using $eq^n(3.56)$ and $eq^n(3.57)$ we get :-

$$_{in}\langle 0|N^{out}_\omega|0\rangle_{in} = \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2 \simeq t\frac{1}{e^{2\pi\omega/\lambda}-1} \tag{3.58}$$

This shows that the near horizon temperature is $T_H = \frac{\lambda}{2\pi}$. One can now proceed as in Section-(3.1) to compute the net flux in (σ^-, σ^+) coordinates at $\mathcal{J}^+_{\mathcal{R}}$ and it will come out to be $\frac{\lambda^2}{48}$ near the horizon. This shows that the net outgoing flux obtained in the previous section is indeed thermal and we will identify this as the Hawking radiation.

3.3 *Carter-Penrose* Diagram

Let us draw the *Carter-Penrose* Diagram for the above process of black hole formation.

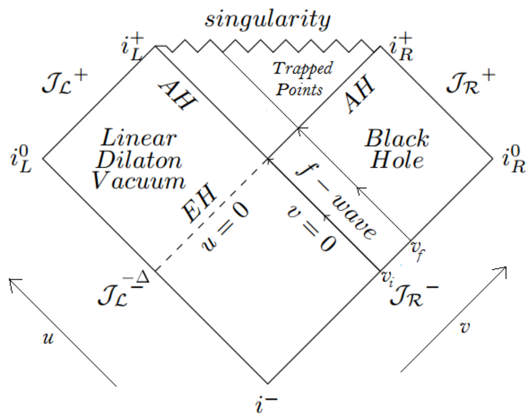


Figure 3.1: *Carter-Penrose* diagram for black hole formation.

Chapter 4

Moving mirrors as a toy model

In this chapter we will see that a moving mirror model is equivalent to studying the 2D dilaton gravity model with a dynamical boundary term.

4.1 An Equivalence

We consider a moving mirror action with $v = x^+$ and $u = x^-$ as, [7] :-

$$S_b = m \int d\tau \sqrt{\partial_\tau x^+ \partial_\tau x^-} - \lambda^2 \int d\tau x^+ \partial_\tau x^- \tag{4.1}$$

where S_b is the boundary action and the matter action being :-

$$S_m = \int_{\pm x^\pm < \pm x^\pm} d^2x \partial_+ f \partial_- f \tag{4.2}$$

Now, the EOMs are :-

$$-\frac{m}{2}\sqrt{\partial_\mp x^\pm} \pm \lambda^2 x^\pm + P_\mp(x^\mp) = 0 \tag{4.3}$$

where, $P_\pm(x^\pm) = \pm \int_{x^\pm}^{\pm\infty} dx^\pm T\pm\pm$. After some computation one can find at the boundary :-

$$\partial_\tau x^+(\lambda^2 x^- - P_+(x^-)) + \partial_\tau x^-(\lambda^2 x^+ + P_-(x^+)) = 0 \tag{4.4}$$

One can get the above equation at the boundary $\phi = \phi_0$ from the condition that $\frac{d}{d\tau}(e^{-2\phi}) = 0$ which is in turn the condition that ϕ is constant along the boundary. Here $e^{-2\phi}$ is as given by $eq^n(2.20)$.

So, this is the equivalence between the moving mirror models and the 2D dilaton gravity models with a boundary. But even then there are some dissimilarities between these models which we will discuss.

Future Prospects

This project work is far from complete and there are works left to be done in this summer which could not be done due to the time constraint of the summer project. These works include :-

1. Some particular examples of black hole formation in the case of 2D dilaton gravity with a dynamical boundary.
2. Studying the Hawking radiation for the 2D dilaton gravity with a dynamical boundary.
3. Studying the quantum dynamics of the moving mirror models using the influence functional approach.
4. Observing the full quantum evolution of the scattering of matter waves off the mirror and seeing whether unitary evolution happens.

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